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Zero subsets, representation of meromorphic functions, and Nevanlinna characteristics in a disc

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Abstract. Let $\Lambda=\{\lambda_k\}$ be a point sequence in the unit disc $\mathbb{D}$ and $N_\Lambda(r)$ the Nevanlinna characteristic of the sequence Λ, $0<r<1$. In terms of the Nevanlinna characteristic $N_\Lambda(r)$ one finds estimates for the slowest possible growth of the characteristic $B(r,|f|)=\max\{|f(z)|:|z|=r\}$ as $r\to 1-0$ in the class of holomorphic functions $f\not\equiv 0$ in $\mathbb{D}$ vanishing on Λ.

Let F be a meromorphic function in $\mathbb{D}$. In terms of the Nevanlinna characteristic function $T(r,F)$ of F one finds estimates for the slowest possible growth of the characteristics $B(r,|g|)$ and $B(r,|h|)$ in the class of pairs of holomorphic functions g and h such that $F=g/h$.

Bibliography: 21 titles.

§ 1. Introduction and statements of results

Let $\mathscr{M}^+(D)$ be the class of positive Borel measures μ in a subdomain D of the complex plane $\mathbb{C}$, and let $\operatorname{supp}\mu$ be the support of a measure $\mu\in\mathscr{M}^+(D)$; let $\mu^{\mathrm{rad}}(r):=\mu\bigl(D(r)\bigr)$, where $D(r):=\{z\in\mathbb{C}:|z|<r\}$, $r>0$; for $r\leqslant 0$ we set by definition $D(r):=\varnothing$.

Let $\Lambda=\{\lambda_k\}$, $k=1,2,\dots$, be a point sequence in the unit disc $\mathbb{D}:=D(1)$ without limit points in $\mathbb{D}$. We associate with Λ an integer-valued measure n_Λ in $\mathbb{D}$ by the formula

$$n_\Lambda(D)=\sum_{\lambda_k\in D}1,\qquad D\subset\mathbb{D},\tag{1.1}$$

so that $n_\Lambda(D)$ is the number of elements of Λ lying in D. In particular, we have $\operatorname{supp}\Lambda:=\operatorname{supp}n_\Lambda$; $\Lambda\subset D$ means by definition that $\operatorname{supp}\Lambda\subset D$; $\lambda\in\Lambda$ (respectively, $\lambda\notin\Lambda$) means the same as $\lambda\in\operatorname{supp}\Lambda$ (respectively, as $\lambda\notin\operatorname{supp}\Lambda$);

$$n_\Lambda^{\mathrm{rad}}(r)=n_\Lambda(D(r))=\sum_{|\lambda_k|<r}1,\qquad r\geqslant 0,\tag{1.2}$$

is the *counting function of the sequence* Λ, that is, $n_\Lambda^{\mathrm{rad}}(r)$ is the number of elements of the sequence lying in the disc $D(r)$. Throughout, we assume, unless otherwise stated, that $0\notin\Lambda$ since this constraint is always easy to eliminate in the current context. We bear in mind this agreement in dealing with the most important

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characteristic N_Λ of the sequence Λ, which we can express in many ways with the use of the standard notation $\log^+ t = \max\{0, \log t\}$:

$$N_\Lambda(r) := \int_0^r n_\Lambda^{\mathrm{rad}}(t)\,\frac{dt}{t} = \int_0^1 \log^+ \frac{r}{t}\,dn_\Lambda^{\mathrm{rad}}(t) = \sum_{\lambda_k \in \Lambda} \log^+ \frac{r}{|\lambda_k|}\,; \qquad (1.3)$$

N_Λ is the *integrated* or the *mean counting function of the sequence* Λ, which in the framework of value distribution theory can be called the *Nevanlinna characteristic of the sequence* Λ.

Following Schwartz [1] (Chapter I, § 4), by positive numbers, functions, measures, and so on, we mean the ones satisfying the relation '$\geqslant 0$'; a similar agreement involving '$\leqslant 0$' holds for negative quantities. If for a function or a map f on a set A we have $f(x) \equiv b$ for $x \in A$, then we write '$f \equiv b$ on A'; otherwise we write '$f \not\equiv b$ on A'. A function f on a subset A of the real axis $\mathbb{R}$ is said to be *increasing* (respectively, *strictly increasing*) if for all t_1, t_2 in A, $t_1 < t_2$, we have $f(t_1) \leqslant f(t_2)$ (respectively, $f(t_1) < f(t_2)$). In similar fashion we distinguish between a *decrease* and a *strict decrease.*

We denote the boundary of a subset D of $\mathbb{C}$ by ∂D.

Let r be a positive number and u a measurable function on $\partial D(r)$ (with respect to arc length) ranging in the extended number line $\mathbb{R} \cup \{\pm\infty\}$. Let us introduce the standard characteristics

$$B(r,u) := \sup\{u(z) : |z| = r\}, \qquad m(r,u) := \frac{1}{2\pi}\int_0^{2\pi} u^+(re^{i\theta})\,d\theta, \qquad (1.4)$$

where by sup we shall mean in general the essential supremum with respect to arc length and, as usual, $u^+ = \max\{0, u\}$ is the positive part of a function u. For a function u defined at zero it is natural to set $B(0,u) := u(0)$ and $m(0,u) := u^+(0)$.

We denote the class of holomorphic functions in a domain $D \subset \mathbb{C}$ by $H(D)$. A function $f \in H(\mathbb{D})$ *vanishes on a sequence* $\Lambda \subset \mathbb{D}$ if the multiplicity of the zero of f at each point $\lambda \in \mathbb{D}$ is not less than the number of occurrences of this point in Λ (we write $f(\Lambda) = 0$); we denote by Zero_f the sequence of zeros of the function f numbered with multiplicities taken into account. If the multiplicity of the zero of f at each point $\lambda \in \mathbb{D}$ is equal to the number of its occurrences in Λ, then Λ is the *zero sequence of* f (we write $\mathrm{Zero}_f = \Lambda$).

We consider the following two related extremal problems.

Problem 1. *In terms of the mean counting function $N_\Lambda(r)$ of a sequence $\Lambda \subset \mathbb{D}$ find an upper estimate for the slowest possible growth as $r \to 1-0$ of the characteristic $B(r,|f|)$ in the class of holomorphic functions $f \not\equiv 0$ in $\mathbb{D}$ vanishing on Λ.*

We denote the class of meromorphic functions in a domain $D \subset \mathbb{C}$ by $M(D)$. Throughout, unless otherwise stated, for $F \in M(\mathbb{D})$ we assume that $F(0) \neq \infty$. This simplifies statements without loss of generality within our frameworks. We denote by $T(r,F)$ the Nevanlinna characteristic function of $F \in M(\mathbb{D})$, that is, if $\Gamma = \{\gamma_k\}$ is the sequence of poles of the function F numbered with multiplicities taken into account, then $T(r,F) := m(r, \log|F|) + N_\Gamma(r)$.

Problem 2. *In terms of the Nevanlinna characteristic function $T(r,F)$ of a meromorphic function $F \in M(\mathbb{D})$ find an estimate of the slowest possible growth of* *the pair of characteristics* $B(r,|g|)$ *and* $B(r,|h|)$ *in the class of pairs of functions* $g,h \in H(\mathbb{D})$ *such that* $F = g/h$.

Special notation. For a real function f and a function $g \geqslant 0$ defined in a left half-neighbourhood of $a \in \mathbb{R}$ the notation $f(r) \lesssim g(r)$ as $r \to a-0$ will mean that there exist constants $C \geqslant 0$ and $\varepsilon > 0$ such that $f(r) \leqslant Cg(r) + C$ for all $a-\varepsilon < r < a$. If $f \geqslant 0$ and $g(r) \lesssim f(r)$ as $r \to a-0$, then we shall write $f(r) \simeq g(r)$ as $r \to a-0$. Thus, the notation $f(r) \lesssim 0$ as $r \to 1-0$ will be equivalent to the upper boundedness of f in a left half-neighbourhood of 1.

Problems 1 and 2 have their origins in R. Nevanlinna's classical theorems on classes of holomorphic and meromorphic functions with bounded characteristic function in a disc [2], [3] (Theorem 6.11), [4] (Theorem V.8), which we formulate below in an unorthodox and slightly abbreviated version to make the relation of our results to their origins more explicit.

Theorem A (R. Nevanlinna). *Let* Λ *be a sequence in* $\mathbb{D}$. *If*

$$N_\Lambda(r) \lesssim 0 \quad \text{as } r \to 1-0, \tag{1.5}$$

then there exists a holomorphic function $f \not\equiv 0$ *in* $\mathbb{D}$ *such that* $f(\Lambda) = 0$ (*moreover*, $\mathrm{Zero}_f = \Lambda$) *and* $\log B(r,|f|) \lesssim N_\Lambda(r)$, $r \to 1-0$.

Theorem B (R. Nevanlinna). *Let* $F \in M(\mathbb{D})$. *If*

$$T(r,F) \lesssim 0 \quad \text{as } r \to 1-0, \tag{1.6}$$

then $F = g/h$ *where* $g,h \in H(\mathbb{D})$ *are functions such that*

$$\log\max\{B(r,|g|), B(r,|h|)\} \lesssim T(r,F), \qquad r \to 1-0.$$

The *order of a meromorphic function* $F \in M(\mathbb{D})$ is by definition the quantity ([4], V.4)

$$\rho := \limsup_{r\to 1-0} \frac{\log T(r,F)}{-\log(1-r)}\,.$$

In particular, for $f \in H(\mathbb{D}) \subset M(\mathbb{D})$ the order can be defined (as in [4] at any rate) by the same relation, that is, in terms of the Nevanlinna characteristic function $T(r,f) = m(r,\log|f|)$ rather than in terms of $B(r,|f|)$. For this reason one can regard Theorem C below as a result close to Theorem B, which is nevertheless not a straightforward generalization of it (see [4], Theorem V.13 as regards the differences).

Theorem C [4], (Theorem V.27). *A meromorphic function in* $\mathbb{D}$ *of order* ρ *can be represented as a ratio of holomorphic functions in* $\mathbb{D}$ *of the same order* ρ.

As concerns similar questions for entire and meromorphic functions in $\mathbb{C}$ and $\mathbb{C}^n$ the reader can consult the survey [5] and the recent paper [6].

For statements of our results here we shall for convenience use the following definition.

Definition 1. We call an arbitrary strictly increasing continuous function

$$d\colon [a,1) \to [0,1) \tag{1.7}$$

defined for some $a \in [0,1)$ and satisfying the constraints

$$t < d(t) < 1 \text{ for all } t \in [a,1) \tag{1.8}$$

a *shift function* (*for the disc* $\mathbb{D}$). We identify two shift functions if there exists a left half-neighbourhood of the point 1 such that the restrictions to it of these functions are the same. We say that a property holds for a shift function if some restriction of it to a left half-neighbourhood of 1 has this property.

We present several graphic realizations of shift functions d, ρ, ϱ in Figs. 1–4 in § 4. In accordance with (1.8), one can always define by continuity a shift function d by setting $d(1) = 1$. Moreover, if a left derivative $d'_-(1)$ of a shift function (1.7) exists at 1 (for example, if d is convex or concave), then it immediately follows by (1.8) that

$$0 \leqslant d'_-(1) \leqslant 1. \tag{1.9}$$

Examples. The following model shift functions are useful to bear in mind:

(1) linear shift functions $d(r) \equiv \alpha r + (1-\alpha)$, where $\alpha \in (0,1)$;

(2) convex shift functions of the form $d(r) \equiv r + (1-r)^\alpha$, where the exponent $\alpha > 1$ is constant, and where $a \geqslant 1 - \alpha^{1/(1-\alpha)}$ in (1.7);

(3) concave shift functions of the form $d(r) \equiv 1 - (1-r)^\alpha$, where the exponent $\alpha > 1$ is constant and where $a > 0$ in (1.7).

Our main results on Problems 1 and 2 presented here and stated as Theorems 1 and 2, respectively, are general: we do not make a priori assumptions similar to (1.5), (1.6) about Nevanlinna characteristics. For special cases of a convex shift function d we announced Theorems 1 and 2 in [6] (§ 3.3, Theorem 2) as analogues of the corresponding results for functions in $\mathbb{C}$.

Theorem 1. *Let Λ be a sequence in $\mathbb{D}$. Then for each convex or concave shift function d there exists a function $f \in H(\mathbb{D})$, $f \not\equiv 0$ on $\mathbb{D}$, vanishing on Λ and satisfying the relation*

$$\log B(r,|f|) \lesssim \frac{1}{d(r)-r}\, N_\Lambda(d(r)) \quad \text{as } r \to 1-0. \tag{1.10}$$

Theorem 2. *Let $F \in M(\mathbb{D})$. Then for each convex or concave shift function d, F has a representation $F = g/h$ in the form of the ratio of functions $g, h \in H(\mathbb{D})$ satisfying the following relation*:

$$\log \max\{B(r,|g|), B(r,|h|)\} \lesssim \frac{1}{d(r)-r}\, T(d(r),F) \quad \text{as } r \to 1-0. \tag{1.11}$$

An increasing coefficient before the Nevanlinna characteristics on the right-hand sides of (1.10) and (1.11) is unavoidable when the characteristics are unbounded. For a linear shift function d, in the case of Nevanlinna characteristics of moderate growth, the coefficient must moreover have the same form as in (1.10) and (1.11), which is directly (see [4], Theorem V.13) or indirectly (see [7], Theorem 2.2) substantiated by known results on meromorphic functions of finite order and on the distribution of zeros of holomorphic functions in weighted spaces on the unit disc $\mathbb{D}$.

The structure of our paper is as follows. In § 2 we present versions of relations between the characteristics (1.4) for subharmonic functions. In § 3 we present the general scheme of the central method of this paper, non-constructive balayage based on the machinery of Jensen measures and functions, and apply results of § 2 to special estimates of the Jensen functions. Auxiliary Propositions 4.1, 4.3, and 4.7 in § 4 on dampening the growth of subharmonic functions are also of independent interest in our opinion. From Proposition 4.7 in § 5 one easily deduced Theorems 1 and 2. In § 5 we also present Corollary 5.1 on a multiplier, an equivalent version of Theorem 1.

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§ 2. Variations on the theme of Harnack's inequality

In this section we find estimates for a function $v \in \mathrm{SH}(D(R))$ in an annulus $D(t) \setminus D(s)$, $0 < s < t < R$, in terms of its characteristics $m(s, v)$ and $m(t, v)$. These estimates are based on a mere combination of Harnack's inequality and the convexity in $\log r$ of the characteristics (1.4) of v, or the two-constant theorem for the harmonic measure in the annulus [8], Theorem 4.3.7.

Let $\mathbb{C}_\infty$ be the extended complex plane (the Riemann sphere) $\mathbb{C} \cup \{\infty\}$. For a subset S of $\mathbb{C}_\infty$ we denote by $\overline{S}$ the closure of S in $\mathbb{C}_\infty$. If $\overline{S}$ is a compact subset of $D \subset \mathbb{C}_\infty$ in the topology induced from $\mathbb{C}_\infty$, then S is precompact in D, and we write $S \Subset D$.

A function v is subharmonic on a subset A of $\mathbb{C}_\infty$ if it is subharmonic on an open subset of $\mathbb{C}_\infty$ containing A (of course, here one takes the quotient by the relation of the equality of two functions on A). Let $\mathrm{SH}(A)$ be the cone of subharmonic functions on $A \subset \mathbb{C}_\infty$ containing also the function $v \equiv -\infty$ on A; $\mathrm{SH}^+(A)$ is the subcone of positive functions in $\mathrm{SH}(A)$.

For $v \in \mathrm{SH}(\overline{D(R)})$ the characteristics $B(r, v)$ and $m(r, v)$ in (1.4) are continuous and convex in $\log r$ on the interval $[0, R]$ ([8], Theorem 2.6.8).[1] Harnack's inequality ([8], Theorem 1.3.1; [9], Theorem 1.18) has the following elementary consequence.

Proposition 2.1. *Let $v \in \mathrm{SH}(\overline{D(t)})$ and assume that $0 \leqslant r \leqslant t$. Then*

$$B(r, v) \leqslant \frac{t + r}{t - r}\, m(t, v). \tag{2.1}$$

For $r = t$ we define the right-hand side (2.1) as its limit as $r \to t - 0$, that is, we set it (here and in similar cases) equal to $+\infty$. We observe that inequality (2.1), which is precise in general, cannot 'feel' a priori upper constraints on the characteristic $m(s, v)$ for $s < r$. The following result is cured of this defect to a certain extent.

Proposition 2.2. *Let $v \in \mathrm{SH}(\overline{D(t)})$ and assume that*

$$0 \leqslant s_1 \leqslant s \leqslant r \leqslant t_1 \leqslant t, \tag{2.2}$$

[1] A function $f(r)$ is convex in $\log r$ if $f(e^x)$ is a convex function of x.

where no three quantities are equal. Then

$$B(r,v) \leqslant \frac{1}{\log(t_1/s_1)}\biggl(\Bigl(\log\frac{t_1}{r}\Bigr)\frac{s+s_1}{s-s_1}\,m(s,v)+\Bigl(\log\frac{r}{s_1}\Bigr)\frac{t+t_1}{t-t_1}\,m(t,v)\biggr). \qquad (2.3)$$

Proof. It follows by Proposition 2.1 that

$$B(s_1,v) \leqslant \frac{s+s_1}{s-s_1}m(s,v), \qquad B(t_1,v) \leqslant \frac{t+t_1}{t-t_1}\,m(t,v). \qquad (2.4)$$

Since $B(r,v)$ is convex in $\log r$, it follows that

$$B(r,v) \leqslant \frac{\log(t_1/r)}{\log(t_1/s_1)}\,B(s_1,v)+\frac{\log(r/s_1)}{\log(t_1/s_1)}\,B(t_1,v).$$

Combining this with (2.4) we obtain (2.3).

Letting s_1 introduced in (2.2) approach zero in (2.3) for fixed $s>0$ we easily see that for $t_1=r$ (2.1) is a special case of (2.3).

Using Proposition 2.2 for $s_1=s^2/r$ and $t_1=\sqrt{rt}$, so that (2.2) holds, we obtain the following easy result.

Corollary 2.1. *Let* $v\in \mathrm{SH}(\overline{D(t)}\,)$ *and assume that it is known that* $v\leqslant 0$ *on the circle* $\partial D(s)$. *If*

$$0 \leqslant s \leqslant r \leqslant \frac{t+s}{2} \leqslant t \leqslant 3s, \qquad (2.5)$$

then

$$B(r,v) \leqslant 2^6\frac{r-s}{(t-s)^2}\,tm(t,v).$$

§ 3. Jensen measures and functions

For a subharmonic function $u\not\equiv -\infty$ in the domain $D\subset \mathbb{C}_\infty$ the positive measure $\nu_u=(1/(2\pi))\Delta u$ (here Δ is the Laplace operator and the inequality holds in the sense of distribution theory) is the Riesz measure of u in D; conversely, for a measure $\nu\in\mathscr{M}^+(D)$, a function u_ν that is subharmonic in D with Riesz measure $\nu_u=\nu$ always exists and is defined up to addition of a harmonic term ([8], Theorem 3.7.9; [9],§ 3.5).

We shall require in what follows several concepts and results, which we formulate here only for the cone $\mathrm{SH}(\mathbb{D})$.

Definition 2 ([10], Introduction; [11], § III.B2; [12], § 1.1; [13], Definition 2). The measure $\mu\in\mathscr{M}^+(\mathbb{D})$ with $\operatorname{supp}\mu\subset\mathbb{D}$ is called a *Jensen measure in* $\mathbb{D}$ if $u(0)\leqslant\int_{\mathbb{D}}u\,d\mu$ for each function $u\in\mathrm{SH}(\mathbb{D})$.

The role of Jensen measures in the proofs of Theorems 1 and 2 is determined by the following result of [14] (the main theorem; see also [15], Theorem 1), which is stated here in a weak form, only for the disc $\mathbb{D}$.

Proposition 3.1. *Let $u \in \mathrm{SH}(\mathbb{D})$ and let $M\colon \mathbb{D} \to \mathbb{R}$ be a continuous function. If there exists a constant C such that*

$$\int_{\mathbb{D}} u\, d\mu \leqslant \int_{\mathbb{D}} M\, d\mu + C \tag{3.1}$$

for all Jensen measures μ in $\mathbb{D}$, then there exists a function $v \in \mathrm{SH}(\mathbb{D})$ such that

$$v(0) \neq -\infty, \qquad u(z) + v(z) \leqslant M(z) \quad \text{for each } z \in \mathbb{D}. \tag{3.2}$$

Below we shall put Proposition 3.1 in another form (as Proposition 3.3) similar to the main theorem of [13] (see also [11], § III.C1, Khabibullin's theorem; [12], Theorem 4.1). The *potential of a Jensen measure μ in $\mathbb{D}$* is the function (see [13], Definition 3; [16], formula (2.9))

$$V_\mu(\zeta) := \int_{\mathbb{D}} \log|z-\zeta|\, d\mu(z) - \log|\zeta|, \qquad \zeta \in \mathbb{D}\setminus\{0\}. \tag{3.3}$$

Definition 3 ([13], Definitions 1 and 1′; [17] a)–c); [18], Definition 2). We call a subharmonic function V in $\mathbb{D}\setminus\{0\}$ a *Jensen function in* $\mathbb{D}$ if the following three conditions hold:

(1) $V(\zeta) \geqslant 0$ for $\zeta \in \mathbb{D}\setminus\{0\}$;

(2) there exists $r \in (0,1)$ such that $V(\zeta) \equiv 0$ for $r \leqslant |\zeta| < 1$ (compact support);

(3)

$$\limsup_{\zeta \to 0} \frac{V(\zeta)}{-\log|\zeta|} \leqslant 1.$$

We always extend Jensen functions V in $\mathbb{D}$ and potentials of Jensen measures (3.3) *by zero onto the complement* $\mathbb{C}_\infty \setminus \mathbb{D}$. Then such functions are subharmonic in $\mathbb{C}_\infty \setminus \{0\}$.

The construction (3.3) of the potentials V_μ of Jensen measures μ in $\mathbb{D}$ establishes a bijection between all Jensen measures in $\mathbb{D}$ and the class of Jensen functions in $\mathbb{D}$ (see [13], Propositions 1.5 and 4.1; [14], Proposition 3.4; [18], the duality theorem). The following generalized Poisson–Jensen formula holds here ([13], Proposition 1.4; [18], Proposition 1.2).

Proposition 3.2. *Let ν be the Riesz measure of a function $u_\nu \in \mathrm{SH}(\mathbb{D})$. Then the following equality holds for each Jensen measure μ in $\mathbb{D}$ with potential V_μ:*

$$\int_{\mathbb{D}} u_\nu\, d\mu = \int_{\mathbb{D}} V_\mu\, d\nu + u_\nu(0). \tag{3.4}$$

In [13] and [18] equality (3.4) is presented in another form, with imposition of the constraint $u_\nu(0) \neq -\infty$; however, in the form (3.4) it holds also without this constraint.

Applying Proposition 3.2 to the left-hand side of (3.1) we obtain the following key result for the proof of our central theorem.

Proposition 3.3. *Let $\nu \in \mathscr{M}^+(\mathbb{D})$ and assume that $0 \notin \operatorname{supp}\nu$, and let $M\colon \mathbb{D} \to \mathbb{R}$ be a continuous function. If*

$$\int_{\mathbb{D}} V_\mu \, d\nu \leqslant \int_{\mathbb{D}} M \, d\mu + C \tag{3.5}$$

holds with some constant C for all Jensen measures μ in $\mathbb{D}$, then for each function $u = u_\nu$ with Riesz measure ν there exists a function $v \in \mathrm{SH}(\mathbb{D})$ such that (3.2) holds.

We now present estimates of Jensen functions in the spirit of § 2.

Proposition 3.4. *Let V be a Jensen function in $\mathbb{D}$ set equal to zero in $\mathbb{C}_\infty \setminus \{0\}$. Then the following universal estimate holds:*

$$B(t,V) \leqslant \frac{t+r}{t-r}\, m(r,V), \qquad 0 \leqslant r \leqslant t < +\infty, \tag{3.6}$$

while if, in addition,

$$\frac{1}{3} \leqslant r \leqslant \frac{2r}{1+r} \leqslant t \leqslant 1 \tag{3.7}$$

then also

$$B(t,V) \leqslant 2^6 \frac{1-t}{(1-r)^2}\, m(r,V). \tag{3.8}$$

Proof. Applying Proposition 2.1 to the inversion

$$v(z) := V\left(\frac{1}{z}\right), \qquad z \in \mathbb{C}, \tag{3.9}$$

of the function V one obtains (3.6).

For $r' := 1/t \geqslant 1$, $s' := 1 =: s$, and $t' := 1/r \leqslant 3$ one obtains from the constraints (3.7) inequalities of the form (2.5):

$$s' = 1 \leqslant r' \leqslant \frac{t'+1}{2} = \frac{t'+s'}{2} \leqslant t' \leqslant 3 = 3s'.$$

The application of Corollary 2.1 to the function v in (3.9) yields

$$B(t,V) = B(r',v) \leqslant 2^6 \frac{r'-1}{(t'-1)^2}\, t' m(t',v) = 2^6 \frac{1-t}{(1-r)^2}\, \frac{r}{t}\, m(r,V), \tag{3.10}$$

which, in view of the condition $r \leqslant t$, leads to (3.8).

Slightly weaker estimates than (3.6) and (3.8) can be combined into a single inequality.

Proposition 3.5. *Let V be a Jensen function in $\mathbb{D}$. Then*

$$B(t,V) \leqslant 2^8 \frac{1-t}{(t-r)(1-r)}\, m(r,V), \qquad 0 \leqslant r \leqslant t \leqslant 1. \tag{3.11}$$

§ 4. Several auxiliary statements

We say that a measure $\lambda \in \mathscr{M}^+(\mathbb{D})$ is *radial* if it has a representation as a tensor product of measures (see [1], Chapter 4, § 8)

$$d\lambda(\zeta) = \frac{1}{2\pi}\, d\theta \otimes d\lambda^{\mathrm{rad}}(t), \qquad \zeta = te^{i\theta} \in \mathbb{D}$$

(here $t \geqslant 0$, $\theta \in [0, 2\pi)$). If a radial measure λ has no strong singularity at the origin (for instance, it is concentrated outside a neighbourhood of the origin), then the function $N_\lambda(|z|)$, $z \in \mathbb{D}$, constructed from the mean counting function (cf. (1.3))

$$N_\lambda(r) := \int_0^r \lambda^{\mathrm{rad}}(t)\, \frac{dt}{t} = \int_0^1 \log^+ \frac{r}{t}\, d\lambda^{\mathrm{rad}}(t), \tag{4.1}$$

is one of the subharmonic functions vanishing at 0 with Riesz measure precisely equal to λ (see [6], 2.3).

Proposition 4.1. *Let $\nu \in \mathscr{M}^+(\mathbb{D})$. Then for each subharmonic function u in $\mathbb{D}$ with Riesz measure ν and for each shift function ρ on $[a, 1)$ described in (1.7), (1.8) there exists a subharmonic function $v \not\equiv -\infty$ in $\mathbb{D}$ such that the following inequality holds*:

$$u(z) + v(z) \leqslant \frac{2^8}{a}\, I_\nu(|z|; \rho) \text{ for each } z \in \mathbb{D}, \tag{4.2}$$

where

$$\begin{gathered} I_\nu(r; \rho) := \int_a^r \int_a^x \frac{1 - \rho(t)}{(\rho(t) - t)(1 - t)}\, d\nu^{\mathrm{rad}}(\rho(t))\, dx \text{ for } a \leqslant r < 1, \\ I_\nu(r; \rho) \equiv 0 \text{ for } 0 \leqslant r < a. \end{gathered} \tag{4.3}$$

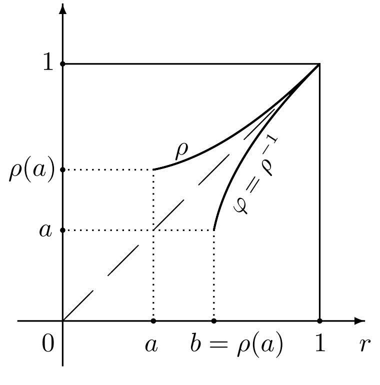

Figure 1

Proof. The shift function $\rho\colon [a, 1) \to [\rho(a), 1)$ possesses the inverse function (see Fig. 1)

$$\varphi := \rho^{-1}\colon [b, 1) \to [a, 1), \qquad b := \rho(a) > a, \tag{4.4}$$

which is also strictly increasing, continuous, and satisfies the conditions

$$\varphi(t) < t \text{ for each } t \in [b,1), \qquad \lim_{t\to 1-0} \varphi(t) = 1. \tag{4.5}$$

In particular, φ is a proper map of $[b,1)$ onto $[a,1)$: the inverse φ-image of a compact subset of $[a,1)$ is a compact subset of $[b,1)$. At this stage, it is more convenient for us to impose the following constraint on the support of ν:

$$\operatorname{supp}\nu \cap D(b) = \varnothing. \tag{4.6}$$

By Proposition 3.5, for $r = \varphi(t) < t \in [b,1)$ we obtain a universal estimate for all Jensen functions V in $\mathbb{D}$:

$$B(t,V) \leqslant 2^8 \frac{1-t}{(t-\varphi(t))(1-\varphi(t))}\, m(\varphi(t),V),$$

where $b \leqslant t < 1$. In view of (1.4) and the positiveness of V, we can write it as follows:

$$V(\zeta) \leqslant 2^8 \frac{1-\varphi(|\zeta|)}{(|\zeta|-\varphi(|\zeta|))(1-\varphi(|\zeta|))}\, \frac{1}{2\pi}\int_0^{2\pi} V(\varphi(|\zeta|)e^{i\theta})\, d\theta \text{ for each } \zeta \in \mathbb{D}\setminus D(b).$$

Hence integrating with respect to $\nu \geqslant 0$ and bearing in mind assumption (4.6) and the compact support of the Jensen functions (condition (2) in Definition 3) we obtain a universal estimate for all Jensen functions V in $\mathbb{D}$:

$$\int V\, d\nu \leqslant 2^8 \int_{b\leqslant|\zeta|<1} \left(\frac{1-\varphi(|\zeta|)}{(|\zeta|-\varphi(|\zeta|))(1-\varphi(|\zeta|))}\, \frac{1}{2\pi}\int_0^{2\pi} V(\varphi(|\zeta|)e^{i\theta})\, d\theta \right) d\nu(\zeta). \tag{4.7}$$

Here we can regard the ‘integral’ right-hand side (without the coefficient 2^8) as an action on the compactly supported (upper semicontinuous outside the origin) function V of the standard extension ([9], Chapter 3) of the continuous linear functional (Radon measure) λ defined on the space $C_0(\mathbb{D}\setminus D(a))$ of compactly supported continuous functions on $\mathbb{D}\setminus D(a)$ by the formula

$$\langle \lambda, f\rangle = \int_b^1 \left(\frac{1-t}{(t-\varphi(t))(1-\varphi(t))}\, \frac{1}{2\pi}\int_0^{2\pi} f(\varphi(t)e^{i\theta})\, d\theta \right) d\nu^{\mathrm{rad}}(t), \tag{4.8}$$

where $\langle \lambda, f\rangle$ is the action of λ on a function $f \in C_0(\mathbb{D}\setminus D(a))$. In other words, the measure λ is obtained from the measure $d\nu^{\mathrm{rad}}(t)$ in $[b,1)$ as the result of three successive transformations:

(1) multiplication of $d\nu^{\mathrm{rad}}(t)$ (see [1], Chapter IV, § 5) by a continuous function $(1-t)/((t-\varphi(t))(1-\varphi(t)))$;

(2) the push-forward to $[a,1)$ of the measure in (1) under the continuous proper map $t \mapsto \varphi(t)$ from $[b,1)$ onto $[a,1)$ (see [1], Chapter IV, § 6);

(3) tensor multiplication of the measure in (2) by $d\theta/(2\pi)$ (see [1], Chapter IV, § 8).

By construction, the measure λ, which we regard now as a set function, is defined on $\mathbb{D}\setminus D(a)$, and after extension by zero to $D(a)$ becomes a radial measure on $\mathbb{D}$. We can now write inequality (4.7) in the following form:

$$\int_{\mathbb{D}} V\, d\nu \leqslant 2^8 \int_{\mathbb{D}} V\, d\lambda. \tag{4.9}$$

By the generalized Poisson–Jensen formula (3.4) applied to the pair λ, N_λ and the Jensen measure μ with potential $V_\mu = V$ we obtain $\int_{\mathbb{D}} V\,d\lambda = \int_{\mathbb{D}} N_\lambda(|z|)\,d\mu(z)$. Hence by (4.9) we obtain a universal inequality for all Jensen measures μ in $\mathbb{D}$:

$$\int_{\mathbb{D}} V_\mu\,d\nu \leqslant \int_{\mathbb{D}} 2^8 N_\lambda(|z|)\,d\mu(z). \tag{4.10}$$

The function $N_\lambda(|z|)$ is continuous by (4.1). Thus, (4.10) means that condition (3.5) of Proposition 3.3 holds with $M(z) = 2^8 N_\lambda(|z|)$ and $C = 0$. Hence for each subharmonic function u with Riesz measure ν there exists a function $v \in \mathrm{SH}(\mathbb{D})$, $v(0) \neq -\infty$, such that

$$u(z) + v(z) \leqslant 2^8 N_\lambda(|z|) \text{ for all } z \in \mathbb{D}. \tag{4.11}$$

For the completion of the proof we must carry out some calculations.

Calculation of the function $N_\lambda(|z|)$. For $x \geqslant 0$ we denote by $\chi_{D(x)}$ and $\chi_{[0,x)}$ the indicators of the disc $D(x)$ in $\mathbb{C}$ and the interval $[0,x)$ in $\mathbb{R}$, respectively. In this notation and in view of (4.8), we obtain

$$\lambda(x) = \langle \lambda, \chi_{D(x)} \rangle = \int_b^1 \left(\frac{1-t}{(t-\varphi(t))(1-\varphi(t))}\, \frac{1}{2\pi} \int_0^{2\pi} \chi_{D(x)}(\varphi(t)e^{i\theta})\,d\theta \right) d\nu^{\mathrm{rad}}(t).$$

The indicator $\chi_{D(x)}$ is radial, therefore we can write the last equality as follows:

$$\lambda(x) = \int_b^1 \frac{1-t}{(t-\varphi(t))(1-\varphi(t))}\, \chi_{[0,x)}(\varphi(t))\,d\nu^{\mathrm{rad}}(t).$$

The substitution $\tau = \varphi(t)$ in the last integral, which, in view of (4.4), (4.5), means that $t = \rho(\tau)$, $\tau \in [a,1)$, leads to the following identities, which hold for all $x \in [a,1)$:

$$\begin{aligned}
\lambda(x) &\equiv \int_a^1 \frac{1-\rho(\tau)}{(\rho(\tau)-\tau)(1-\tau)}\, \chi_{[0,x)}(\tau)\,d\nu^{\mathrm{rad}}(\rho(\tau)) \\
&\equiv \int_a^x \frac{1-\rho(t)}{(\rho(t)-t)(1-t)}\,d\nu^{\mathrm{rad}}(\rho(\tau)).
\end{aligned}$$

We can now write the definition (4.1) of N_λ as follows:

$$\begin{aligned}
N_\lambda(r) &= \int_a^r \lambda(x)\,\frac{dx}{x} = \int_a^r\!\!\int_a^x \frac{1-\rho(t)}{(\rho(t)-t)(1-t)}\,d\nu^{\mathrm{rad}}(\rho(\tau))\,\frac{dx}{x} \\
&\leqslant \frac{1}{a}\int_a^r\!\!\int_a^x \frac{1-\rho(t)}{(\rho(t)-t)(1-t)}\,d\nu^{\mathrm{rad}}(\rho(\tau))\,dx, \qquad a \leqslant r < 1, \\
N_\lambda(r) &\equiv 0, \qquad 0 \leqslant r < a.
\end{aligned}$$

In view of (4.11), we now obtain (4.2) under the constraint (4.6).

We represent an arbitrary measure ν as a sum of measures $\nu = \nu_0 + \nu_1$, where ν_0 is the restriction of ν to the disc $D(b)$. Correspondingly, we represent $u = u_\nu$ as a sum of subharmonic functions:

$$u(z) = \int_{|\zeta|<b} \log \frac{|z-\zeta|}{2}\, d\nu(\zeta) + u_{\nu_1} =: u_0 + u_1,$$

where u_0 and $u_1 := u_{\nu_1}$ are subharmonic functions with Riesz measures ν_0 and ν_1, respectively. By construction $u_0 \leqslant 0$ on $\mathbb{D}$ and $\operatorname{supp}\nu_1 \subset \mathbb{D} \setminus D(b)$. It is now sufficient to apply the already established version of the proposition to u_1 with measure ν_1 satisfying a condition of the form (4.6).

The following very special case of a multidimensional result of [19] (see also [15], Lemma 2.2) enables us to proceed from the previous purely 'subharmonic' result to its holomorphic version.

Proposition 4.2 ([19], Proposition 9.1)[2]. *If $u, v \in \mathrm{SH}(\mathbb{D})$, $v \not\equiv -\infty$, and for some continuous function M the inequality $u(z) + v(z) \leqslant M(z)$ holds for all $z \in \mathbb{D}$, then there exists a holomorphic function $q \not\equiv 0$ in $\mathbb{D}$ such that at each point $z \in \mathbb{D}$ for each $\sigma \in (0, 1-|z|)$ one has $u(z) + \log|q(z)| \leqslant \sup\{M(\zeta) : |z-\zeta| < \sigma\} - \log\sigma$.*

The following result immediately follows from Propositions 4.1 and 4.2 in view of the increase in r of the function $I_\nu(r;\rho)$ in (4.3).

Proposition 4.3. *Let $\nu \in \mathscr{M}^+(\mathbb{D})$. Then for each subharmonic function u in $\mathbb{D}$ with Riesz measure ν and for each shift function $\rho\colon [a,1) \to [0,1)$ there exists a holomorphic function $q \not\equiv 0$ in $\mathbb{D}$ such that the following estimate holds:*

$$u(z) + \log|q(z)| \leqslant \inf_{0<\sigma<1-|z|} \left(\frac{2^8}{a}\, I_\nu(|z|+\sigma;\rho) + \log\frac{1}{\sigma} \right) \text{ for all } z \in \mathbb{D}, \tag{4.12}$$

where $I_\nu(r;\rho)$ is defined in (4.3).

In view of inequalities (4.2) and (4.12) from Propositions 4.1 and 4.3, the following fairly simple estimates of the double integral (4.3) seem appropriate. Throughout the rest of this section we consider an arbitrary measure

$$\nu \in \mathscr{M}^+(\mathbb{D}), \qquad 0 \notin \operatorname{supp}\nu, \tag{4.13}$$

and an arbitrary shift function ρ. It is convenient to express the corresponding double integral in (4.3) in the following form:

$$I_\nu(r;\rho) = \int^r\!\!\int^x \frac{1-\rho(t)}{(\rho(t)-t)(1-t)}\, d\nu^{\mathrm{rad}}(\rho(t))\, dx, \tag{4.14}$$

without indicating the lower limits of integration since the concrete value of $a < 1$, that is, the left end-point of the domain of definition of the shift function ρ is of no importance for the estimates that follow.

We shall require elementary consequences of the definition of the mean counting function (4.1), some of which were proved in [20] (we use the special notation of § 1).

[2] In [19], by definition we do not include the function identically equal to $-\infty$ in the cone of (pluri)subharmonic functions.

Proposition 4.4. *The measure ν in (4.13) satisfies the following relations.*

$$\int^r \nu^{\mathrm{rad}}(t)\,dt \leqslant N_\nu(r) \lesssim \int^r \nu^{\mathrm{rad}}(t)\,dt, \quad \text{as } r\to 1-0, \tag{4.15i}$$

$$\nu^{\mathrm{rad}}(r) \leqslant \frac{1}{d}\,N_\nu(r+d), \qquad 0\leqslant r<r+d<1, \tag{4.15d}$$

$$N_\nu(r) \simeq \int^r (r-t)\,d\nu^{\mathrm{rad}}(t), \qquad \text{as } r\to 1-0. \tag{4.15l}$$

In the derivation of estimates for the double integral (4.14) it is reasonable to consider separately convex and concave shift functions $\rho(t)$ for t sufficiently close to 1.

(1) *The case of a convex shift function ρ.* In view of the convexity of $\rho(t)$, the left derivative ρ'_- has the following properties for some $\varepsilon>0$ and all t sufficiently close to 1 from the left (see Fig. 1):

$$\varepsilon \leqslant \rho'_-(t) \leqslant \rho'_-(1) \leqslant 1, \qquad \rho(t)-t \ \text{ is a decreasing function.} \tag{4.16}$$

Hence

$$I_\nu(r;\rho) \leqslant \int^r\!\!\int^x \frac{1-t}{(\rho(t)-t)(1-t)}\,d\nu^{\mathrm{rad}}(\rho(t))\,dx \leqslant \int^r \frac{\nu^{\mathrm{rad}}(\rho(x))}{\rho(x)-x}\,dx. \tag{4.17}$$

We can write the last integral in the following form:

$$\int^r \frac{1}{\rho'_-(x)(\rho(x)-x)}\,d\Bigl(\int^x \nu^{\mathrm{rad}}(\rho(t))\,d\rho(t)\Bigr).$$

Hence we can continue the estimate (4.17) by taking account of (4.16):

$$I_\nu(r;\rho) \lesssim \frac{1}{\rho(r)-r}\int^r \nu^{\mathrm{rad}}(\rho(t))\,d\rho(t) = \frac{1}{\rho(r)-r}\int^{\rho(r)} \nu^{\mathrm{rad}}(t)\,dt,$$

which in view of (4.15i), yields the following result.

Proposition 4.5. *If the shift function ρ is convex, then*

$$I_\nu(r;\rho) \lesssim \frac{1}{\rho(r)-r}\,N_\nu(\rho(r)) \quad \textit{as } r\to 1-0. \tag{4.18}$$

(2) *The case of a concave shift function ρ.* Since $\rho(t)$ is concave, the lower estimate $\rho(t)-t\geqslant\varepsilon(1-t)$ follows easily for some $\varepsilon>0$ and for t sufficiently close to 1. Hence (4.14) has the following upper estimate:

$$I_\nu(r;\rho) \lesssim \int^r\!\!\int^x \frac{1-\rho(t)}{(1-t)^2}\,d\nu^{\mathrm{rad}}(\rho(t))\,dx \quad \text{as } r\to 1-0,$$

and, passing to a slightly 'cruder' inequality,

$$I_\nu(r;\rho) \lesssim \int^r \Bigl(\int^x (1-\rho(t))\,d\nu^{\mathrm{rad}}(\rho(t))\Bigr)\,d\frac{1}{1-x} \quad \text{as } r\to 1-0.$$

'Multiplying out' the inner integral after increasing the upper limit of integration in it to r yields

$$I_\nu(r;\rho) \lesssim \frac{1}{1-r}\int^r (1-\rho(t))\,d\nu^{\mathrm{rad}}(\rho(t)) \quad \text{as } r\to 1-0, \tag{4.19}$$

where the integral on the right-hand side satisfies the relation

$$\int^r (1-\rho(t))\,d\nu^{\mathrm{rad}}(\rho(t)) = \int^{\rho(r)} (1-t)\,d\nu^{\mathrm{rad}}(t) \leqslant \int_0^{\rho(r)} (1-t)\,d\nu^{\mathrm{rad}}(t). \tag{4.20}$$

For the last integral, in view of (4.15l), we obtain

$$\begin{aligned}\int_0^{\rho(r)} (1-t)\,d\nu^{\mathrm{rad}}(t) &= \int_0^{\rho(r)} (\rho(r)-t)\,d\nu^{\mathrm{rad}}(t) + (1-\rho(r))\nu^{\mathrm{rad}}(\rho(r))\\ &= N_\nu(\rho(r)) + (1-\rho(r))\nu^{\mathrm{rad}}(\rho(r)).\end{aligned} \tag{4.21}$$

For $d=(1-\rho(r))/2$ it follows from (4.15d) that

$$\nu^{\mathrm{rad}}(\rho(r)) \leqslant \frac{1}{d}\,N_\nu(\rho(r)+d) = \frac{2}{1-\rho(r)}\,N_\nu\biggl(\frac{1+\rho(r)}{2}\biggr).$$

Applying the last inequality to (4.21) we extend the estimate (4.19):

$$I_\nu(r;\rho) \lesssim \frac{1}{1-r}\biggl(N_\nu(\rho(r)) + 2N_\nu\biggl(\frac{1+\rho(r)}{2}\biggr)\biggr) \quad \text{as } r\to 1-0.$$

Using our special notation, from the last relation we arrive at the following result.

Proposition 4.6. *If the shift function ρ is concave, then the double integral in* (4.14) *has the estimate*

$$I_\nu(r;\rho) \lesssim \frac{1}{1-r}\,N_\nu\biggl(\frac{1+\rho(r)}{2}\biggr) \quad \textit{as } r\to 1-0. \tag{4.22}$$

Proposition 4.7. *Let $\nu\in\mathscr{M}^+(\mathbb{D})$. Then for each subharmonic function u in $\mathbb{D}$ with Riesz measure ν, for each convex or concave shift function d there exists a holomorphic function $q\not\equiv 0$ in $\mathbb{D}$ such that*

$$u(z)+\log|q(z)| \lesssim \frac{1}{d(r)-r}\,N_\nu(d(r)) \quad \textit{as } r\to 1-0. \tag{4.23}$$

To deduce this result from Proposition 4.3 we shall require three lemmas describing the construction, for shift functions in $\mathbb{D}$ of various kinds, of certain special minorants, which are also shift functions.

Lemma 4.1. *If d is a convex shift function, then*

$$\rho(r) := \frac{1}{2}(r+d(r))$$

is a convex shift function, which for

$$0 < \sigma := \frac{1}{2}(d(r) - r) < \frac{1}{2}(1 - r) \tag{4.24}$$

has the following properties:

$$\rho(r + \sigma) \leqslant d(r), \qquad \frac{1}{\rho(r+\sigma) - (r+\sigma)} \leqslant \frac{4}{d(r) - r}\,. \tag{4.25}$$

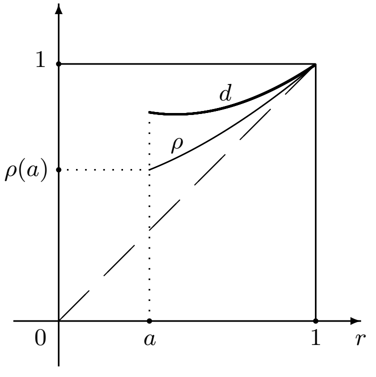

Figure 2

Proof of the lemma (see Fig. 2). By construction ρ is a convex shift function and $\sigma > 0$ in (4.24) satisfies the constraints in (4.24). By the definition of a convex function we obtain

$$d(r + \sigma) \leqslant \left(1 - \frac{\sigma}{1 - r}\right) d(r) + \frac{\sigma}{1 - r}\,.$$

Hence for the function ρ we have

$$\rho(r+\sigma) = \frac{1}{2}(r+\sigma) + \frac{1}{2}\,d(r+\sigma) \leqslant \frac{1}{2}(r+\sigma) + \frac{1}{2}\left(1 - \frac{\sigma}{1-r}\right) d(r) + \frac{1}{2}\,\frac{\sigma}{1-r}\,. \tag{4.26}$$

In what follows, for fixed r we set for convenience $d := d(r) > r$. Assume that the first inequality in (4.25) fails. Then in view of (4.26), we have, a fortiori,

$$(r + \sigma) + \left(1 - \frac{\sigma}{1 - r}\right) d + \frac{\sigma}{1 - r} > 2d,$$

so that $\sigma - \sigma d/(1 - r) + \sigma/(1 - r) > d - r$, which after multiplying out σ yields

$$\sigma\,\frac{(1 - r) + (1 - d)}{1 - r} > d - r.$$

Hence, in view of the inequalities $0 < 1 - d < 1 - r$, we obtain $\sigma > (d - r)/2$, which contradicts our choice of σ in (4.24). In other words, we have established the first inequality in (4.25). The second inequality in (4.25) is trivial.

Before passing to concave shift functions d, we point out that we can refine relation (1.9) for such functions:

$$0 \leqslant d'_-(1) < 1. \tag{4.27}$$

Lemma 4.2. *Let* $d\colon [a,1) \to [0,1)$ *be a concave shift function for* $\mathbb{D}$ *and assume that its left derivative* $d'_-(1)$, *in addition to* (4.27), *satisfies the condition*

$$d'_-(1) \geqslant \frac{1}{4}\,. \tag{4.28}$$

Then there exists a convex (*linear*) *shift function* ϱ *such that*

$$\varrho(r) \leqslant d(r) \text{ for all } r \in [a,1) \tag{4.29}$$

and, in addition,

$$\frac{1}{\varrho(r)-r} \leqslant \frac{c}{d(r)-r} \quad \text{for all } r \in [a,1), \quad c = \frac{3}{4}\,\frac{1-a}{d(a)-a}\,. \tag{4.30}$$

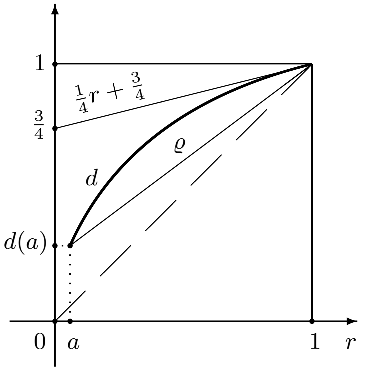

Figure 3

Proof of the Lemma (see Fig. 3). It is sufficient to take for the shift function ϱ the linear function with graph coinciding with the line interval joining the initial point of the graph of d (that is, the point $(a, d(a))$) with the point $(1,1)$. By the concavity of d we immediately obtain (4.29). Direct calculations yield

$$\varrho(r) - r \equiv \frac{d(a)-a}{1-a}(1-r). \tag{4.31}$$

The combination of (4.28) with the properties of the concavity and the increase of d shows that the graph of d lies no higher that the straight line with slope $1/4$ passing through the point $(1,1)$. With (4.31) taken into account this yields the required inequality (4.30).

Lemma 4.3. *Let d be a concave shift function for $\mathbb{D}$ and assume that $d'_-(1)$, its left derivative at* 1, *satisfies the following condition*;

$$d'_-(1) < \frac{1}{4}\,. \tag{4.32}$$

Then

$$\rho(r) := 2d\left(\frac{4r-1}{3}\right) - 1 \tag{4.33}$$

is a concave shift function that for

$$\sigma := \frac{1-r}{4} \tag{4.34}$$

has the following properties:

$$\frac{1+\rho(r+\sigma)}{2} \leqslant d(r), \qquad \frac{1}{1-(r+\sigma)} \leqslant \frac{4/3}{d(r)-r}\,. \tag{4.35}$$

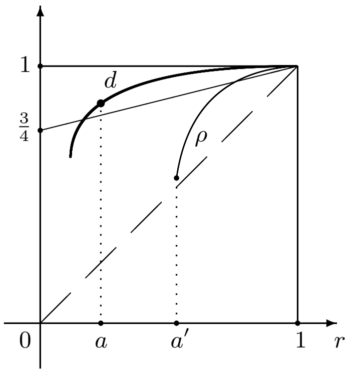

Figure 4

Proof of the lemma (see Fig. 4). The concavity and the strict increase of the function ρ in (4.33), and also the upper bound $\rho < 1$ are obvious from the construction. To verify that ρ is a shift function (see Definition 1, condition (1.8)) it is important to show that for r sufficiently close to 1 we have a lower bound $\rho(r) > r$. Condition (4.32) for the left half-tangent at $(1,1)$ to the graph of d in combination with its increase means that for some $a < 1$ we have the lower bound $d(x) > x/4 + 3/4$ for all $x \in [a,1)$. Applying this estimate to (4.33) we obtain

$$\rho(r) = 2d\left(\frac{4r-1}{3}\right) - 1 > 2\left(\frac{1}{4}\,\frac{4r-1}{3} + \frac{3}{4}\right) - 1 = \frac{2r+1}{3} > r \tag{4.36}$$

provided that $(4r-1)/3 \in [a,1)$, that is, for $r \geqslant (3a+1)/4 =: a'$ (since $a < 1$, we also have $a' < 1$). In combination with (4.36) this proves condition $\rho(r) > r$ on the non-empty interval $[a',1)$, so that ρ is a concave shift function.

We verify (4.35) with σ as in (4.34) by a straightforward substitution and calculations.

We can now prove Proposition 4.7

Proof of Proposition 4.7. We partition the proof into three cases, in accordance with above-proved Lemmas 4.1–4.3. We assume throughout that ν is a non-trivial measure.

(1) *The case of a convex shift function d.* We take a convex shift function ρ as in Lemma 4.1. By Proposition 4.3 there exists a function $q \in H(\mathbb{D})$, $q \not\equiv 0$, such that

$$u(z) + \log|q(z)| \leqslant C\, I_\nu(r+\sigma;\rho) + \log\frac{1}{\sigma}\,, \qquad |z| = r, \tag{4.37}$$

for each $\sigma \in (0, 1-r)$, where the constant C is independent of r and σ. By Proposition 4.5 we have the estimate

$$I_\nu(r+\sigma;\rho) \leqslant C_1 \frac{1}{\rho(r+\sigma)-(r+\sigma)}\, N_\nu(\rho(r+\sigma)) + C_1 \tag{4.38}$$

for r sufficiently close to 1 from the left, with constant C_1 independent of r and σ lying in $(0, 1-r)$. If we also select σ for each r as in Lemma 4.1, so that (4.24) holds, then, in view of (4.25), it follows by (4.38) and (4.37) that

$$u(z) + \log|q(z)| \leqslant CC_1 \frac{4}{d(r)-r}\, N_\nu(d(r)) + CC_1 + \log\frac{2}{d(r)-r}\,. \tag{4.39}$$

The growth in the neighbourhood of 1 of the last ‘logarithmic’ term is obviously ‘dampened’ by the growth of the first term of the right-hand side of (4.39), provided that one increases the constant coefficient of the first term. In this way we obtain (4.23) and prove Proposition 4.7 in the case under consideration.

(2) *The case of a concave shift function d satisfying* (4.28). In view of the estimate (4.30), this intermediate case is covered by the version of Proposition 4.7 just proved, with a convex shift function once one takes (in accordance with Lemma 4.2) a convex (even a linear) shift function $\varrho \leqslant d$.

(3) *The case of a concave shift function d satisfying* (4.32). We take a concave shift function ρ as in Lemma 4.3, that is, in accordance with (4.33). By Proposition 4.3 there exists a function $q \in H(\mathbb{D})$, $q \not\equiv 0$, such that (4.37) holds. Then by Proposition (4.6) we have the estimate

$$I_\nu(r+\sigma;\rho) \leqslant C_1 \frac{1}{1-(r+\sigma)}\, N_\nu\left(\frac{1+\rho(r+\sigma)}{2}\right) + C_1 \tag{4.40}$$

with constant C_1 independent of r and σ. Taking $\sigma := (1-r)/4$ as in (4.34) and bearing in mind the final inequalities (4.35) of Lemma 4.3, in view of (4.40), we obtain by (4.37)

$$u(z) + \log|q(z)| \leqslant CC_1 \frac{4/3}{d(r)-r}\, N_\nu(d(r)) + CC_1 + \log\frac{4}{1-r}\,, \tag{4.41}$$

where the growth in the neighbourhood of 1 of the last ‘logarithmic’ term is obviously ‘dampened’ by the growth of the first term of the right-hand side of (4.41), provided that one increases the constant coefficient of the first term. In this way we obtain (4.23).

§ 5. Proofs of Theorems 1 and 2

Proof of Theorem 1. Recall that $0 \notin \Lambda$ by agreement. We consider a holomorphic function f_Λ in $\mathbb{D}$ with sequence of zeros coinciding with Λ: $\mathrm{Zero}_{f_\Lambda} = \Lambda$, $0 \notin \Lambda$, $f_\Lambda(0) \neq 0$. Let $u = \log|f_\Lambda|$ be the subharmonic function with Riesz measure n_Λ. By Proposition 4.7 (with measure n_Λ in place of ν) there exists a function $q \in H(\mathbb{D})$, $q \not\equiv 0$, such that

$$\log|f_\Lambda(z)| + \log|q(z)| \lesssim \frac{1}{d(r) - r} N_{n_\Lambda}(d(r)) \quad \text{as } |z| = r \to 1 - 0.$$

Then $f = f_\Lambda q$ is the required function because by the definitions and notation (1.1), (1.3), (4.1) we obtain $N_{n_\Lambda} = N_\Lambda$ and also $f(\Lambda) = 0$.

Proof of Theorem 2. For $F \in M(\mathbb{D})$, $F(0) \neq \infty$, let $F = g_F/h_F$ be the representation of the function as the ratio of holomorphic functions g_F and h_F in $\mathbb{D}$ without common zeros; $h_F(0) \neq 0$. Then we can also define the Nevanlinna characteristic function $T(r, F)$ as the integral (see [21], after Theorem 4.1)

$$T(r, F) = \frac{1}{2\pi} \int_0^{2\pi} u^F(re^{i\theta})\, d\theta, \qquad u^F(z) = \max\bigl\{\log|g_F(z)|, \log|h_F(z)|\bigr\}. \tag{5.1}$$

By construction u^F is a subharmonic function. By our agreement that $F(0) \neq \infty$ we have $u^F(0) \neq -\infty$. Moreover, by the classical Poisson–Jensen formula ([8], § 4.5; [9], Theorem 3.14) and in view of (5.1), for each $r \in (0, 1)$ we obtain

$$T(r, F) = \frac{1}{2\pi} \int_0^{2\pi} u^F(re^{i\theta})\, d\theta = N_\nu(r) + u^F(0), \tag{5.2}$$

where ν is the Riesz measure of $u^F \in \mathrm{SH}(\mathbb{D})$. We can assume without loss of generality that ν is non-trivial (otherwise we could take g_F and h_F identically constant). By Proposition 4.7 there exists a function $q \in H(\mathbb{D})$, $q \not\equiv 0$, such that, in view of (5.1) and (5.2), we have the relations

$$\begin{aligned} \max\bigl\{\log|g_F q|(z), \log|h_F q|(z)\bigr\} = u^F(z) + \log|q(z)| &\lesssim \frac{1}{d(r) - r} N_\nu(d(r)) \\ &\lesssim \frac{1}{d(r) - r} T(d(r), F) \quad \text{as } |z| = r \to 1 - 0. \end{aligned}$$

We see that the pair of functions $g := g_F q$ and $h := h_F q$ satisfies conditions (1.11) and can be used for the representation of $F = g_F/h_F = (g_F q)/(h_F q) = g/h$.

We now complement Theorem 1 by its version in the form of a multiplier theorem (cf. the subject of [11]).

Corollary 5.1 (on a multiplier). *Let $f \in H(\mathbb{D})$. Then for each convex or concave shift function d there exists a holomorphic multiplier function $q \not\equiv 0$ in $\mathbb{D}$ such that*

$$\log B(r, |fq|) \lesssim \frac{1}{d(r) - r} \frac{1}{2\pi} \int_0^{2\pi} \log|f(d(r)e^{i\theta})|\, d\theta \quad \textit{as } |z| = r \to 1 - 0. \tag{5.3}$$

Proof. This result is non-trivial only for $f \not\equiv 0$ in $\mathbb{D}$. We can assume without loss of generality that $f(0) \neq 0$. Let $\Lambda = \mathrm{Zero}_f$, $0 \notin \Lambda$. By Theorem 1 there exists a non-trivial function $F \in H(\mathbb{D})$ such that (1.10) holds with F in place of f and, at the same time, $F(\Lambda) = 0$. The last equality means that $q = F/f$ is a non-trivial holomorphic function in $\mathbb{D}$. Moreover, by the Poisson–Jensen formula one can replace the characteristic $N_\Lambda(d(r))$ in (1.10) by the mean value of the function $\log|f|$ on the circle $\partial D(d(r))$, which yields (5.3).

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