On the representation of meromorphic functions

Bulat N. Khabibullin
Bashkir State University (Ufa, Russia)
E-mail: khabib-bulat@mail.ru Web-site: www.bashedu.ru/khabib-bulat

Let Ω be a domain in the complex plane $\mathbb{C}$, $0 \in \Omega$. Denote by $H(\Omega)$ the space of holomorphic functions on Ω. Let $H(\Omega; M) := \left\{ f \in H(\Omega)\colon \sup\limits_{z\in\Omega} \dfrac{|f(z)|}{\exp M(z)} < \infty \right\}$ where M is a subharmonic continuous function on Ω with Riesz measure ν_M. Let f/g be a meromorphic function on Ω, $f, g \in H(\Omega)$. Here we propose a new general method of attack on the following problem: *under what conditions there are functions $f_0, g_0 \in H(\Omega; M)$ without common zeros such that $f/g = f_0/g_0$?*

Let $h\colon \Omega^2 \to (-\infty, +\infty)$ be a Borel-measurable function which is locally bounded, and, for every fixed point $w \in \Omega$, the function $h(w, \cdot)$ is *harmonic on* Ω. The function $k(w, z) := \log|w - z| + h(w, z)$, $(w, z) \in \Omega^2$, is *a subharmonic kernel on* Ω. Such a kernel k on Ω is *suitable for* M if for any $z \in \Omega$ there are a relatively compact subdomain $D_z \ni z$ of Ω and a function $q \in L^1(\Omega \setminus D_z, d\nu_M)$ such that $\bigl|k(w, \zeta)\bigr| \leq q(w)$, $\forall w \in \Omega \setminus D_z$, $\forall \zeta \in D_z$. In that case $M(z) = \int_\Omega k(w, z)\, d\nu_M(w) + H(z)$, $z \in \Omega$, where H is harmonic on Ω. Let $a^+ := \max\{0, a\}$, $\int_\Omega \bigl(k(w, 0) - k(w, z)\bigr)^+ d\nu_M(w) \leq Q(z)$ for all $z \in \Omega$ where Q is continuous on Ω.

Main Theorem [Kh05]. *Let $\Omega \subset \mathbb{C}$ be a n-connected domain, M be a continuous subharmonic function on Ω, k be a suitable subharmonic kernel on Ω for M, and $f, g \in H(\Omega)$. Suppose that $f, g \in H(\Omega; M)$ or there is a constant C such that $\int \log \max\{|f|, |g|\}\, d\omega_D(0, \cdot) \leq \int M\, d\omega_D(0, \cdot) + C$ for all connected unions $D \ni 0$ of finite numbers of relatively compact open disks of Ω where $\omega_D(0, \cdot)$ is the harmonic measure for D at 0. Then there are a constant $b < n - 1$ and $f_0, g_0 \in H\bigl(\Omega; M + Q + b^+(\log|\cdot|)^+\bigr)$ without common zeros such that $f/g = f_0/g_0$ where Q is defined above. If the interior of $\mathbb{C} \setminus \Omega$ is nonempty, then $b^+ = 0$.*

Let $\mathbb{D}$ be the open unit disk. If M is a function on $[0, 1)$, then we extend $M(z) \equiv M(|z|)$ for all $z \in \mathbb{D}$ and $H(\mathbb{D}; M)$ is well defined.

Theorem [Kh04; Theorem 0.1]. *Let $M \geq 0$ be a increasing convex function of* $\log$ *on $[0, 1)$ and $\int_r^{1^-} (1 - t)\, dM(t) = O(1 - r)$ as $r \to 1^-$. If $f, g \in H(\mathbb{D}; M)$, then there are $f_0, g_0 \in H(\mathbb{D}; M)$ without common zeros such that $f/g = f_0/g_0$.*

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References

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