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Closed Submodules of Holomorphic Functions with Two Generators[*]

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Abstract. Let I be a closed submodule over a polynomial ring in a space of holomorphic functions on a domain in the complex plane. We establish sufficient conditions under which I is generated by two functions or two special submodules. As a corollary, it follows from these results that if an invariant subspace $W \subset C^\infty(a,b)$ (with respect to the differentiation operator) admits spectral synthesis, then it is the solution space of a system of two homogeneous convolution equations.

Key words: ???

Introduction

In this paper, we mainly use the terminology in [1–3] and, for the notions of functional analysis, in [4].

In what follows, Ω is a domain in the complex plane $\mathbb{C}$, $H(\Omega)$ is the space of all holomorphic functions in Ω, and $\mathscr{P} \subset H(\Omega)$ is some *separable locally convex space over* $\mathbb{C}$, which, for brevity, will be referred to as a *space*. Everywhere below, it will be assumed that the *topology of the space* $\mathscr{P}$ *majorizes the topology of pointwise* or *simple convergence on* Ω.

If the space $\mathscr{P}$ is a *topological module over the polynomial ring* $\mathbb{C}[z]$, $z \in \mathbb{C}$, or a *topological algebra*, then, accordingly, $\mathscr{P}$ will be simply called a *module* or an *algebra* for brevity. Thus, in what follows, the space $\mathscr{P}$ is a module if it is closed and continuous with respect to the multiplication by the independent variable z.

A subspace I in the module $\mathscr{P}$ is a *submodule*[**] over $\mathbb{C}[z]$ if $pf \in I$ for all $p \in \mathbb{C}[z]$ and $f \in \mathscr{P}$. In particular, if $\mathscr{P}$ is an algebra containing all polynomials, then every ideal in $\mathscr{P}$ is a submodule.

A closed submodule I in the module $\mathscr{P}$ is said to be (topologically) n-*generated*, $n \in \mathbb{N}$, if there is a set of n elements $g_1, \dots, g_n \in I$ such that I coincides with the closure in $\mathscr{P}$ of the set of elements of the form $p_1 g_1 + \dots + p_n g_n$, where $p_1, \dots, p_n \in \mathbb{C}[z]$. In this case, $g_1, \dots, g_n$ are the *functions generating the submodule* I, and the submodule I itself is denoted by $\bar{I}(g_1, \dots, g_n)$. A submodule is said to be *principal* if it is 1-generated. More generally, a closed submodule $I \subset \mathscr{P}$ is said to be (topologically) *generated by submodules* $I_1, \dots, I_n$ if it is the closure of the set of elements of the form $i_1 + \dots + i_n$, where $i_1 \in I_1, \dots, i_n \in I_n$.

In this paper, on the basis of the results of I. F. Krasichkov-Ternovskii's foundational investigations on the problem of local description of closed submodules in spaces of holomorphic function of one variable, we establish conditions under which a *closed submodule* I *in the module* $\mathscr{P}$ *is* 2-*generated* or *is generated by two special submodules*.

In the case $\Omega = \mathbb{D}$, where $\mathbb{D}$ is a unit disk, the topics in the present paper are close to the numerous descriptions (that appeared after the publication of A. Beurling's famous theorem on the description of z-invariant subspaces in the Hardy class H^2) of algebras in $H(\mathbb{D})$ whose every closed ideal is principal. (See the surveys by Nikolskii [5, Secs. 7 and 9–13] and [6] and the references there.) Among the series of works in this investigation direction by F. A. Shamoyan and his disciples, we only mention [7, 8].

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[**] In his connection, the term *z-invariant* or *translation-invariant subspace* is also widespread [5, 6].

In weighted algebras of entire functions determined by *radial majorants*, every closed ideal is almost always 2-generated (see [9]) and, for slowly increasing majorants, even 2-generated in the algebraic sense [10]. In this situation, there can be closed ideals that are not principal, e.g., in algebras of integer-order entire functions of finite type. Abuzyarova (see [11–13]) established recently that, in the case of weighted modules of entire functions singled out by some conditions on the indicator, every closed submodule admitting a local description (for the related definitions, see Sec. 1) is 2-generated. Our results include the above-mentioned situations and also some very wide classes (see Sec. 5) of modules of holomorphic functions in domains belonging to $\mathbb{C}$. Applications of the main results in this paper (Theorems 1 and 2) to the problem of spectral synthesis are discussed in Sec. 5 for the simple, but rather meaningful case of the space $C^\infty(a,b)$ of infinitely differentiable functions on an interval $(a,b)\subset\mathbb{R}$. By Theorem 4, every differentiation-invariant closed subspace $W\subset C^\infty(a,b)$ such that the linear span of all exponential monomials contained in W is dense in it can be defined as a solution space of at most two homogeneous convolution equations.

The unquestionable connection of the problem under consideration with corona theorems and with weakly invertible elements is not discussed here. The simpler case in which $\mathscr{P}$ is an algebra and in which only closed ideals instead of closed submodules are considered is touched upon in this paper merely episodically since it was considered rather completely in [9]. Some of the results obtained in [9] and in this paper were partially announced earlier. (See [14, Theorems A and M].)

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1. Main Notions and Results

In what follows, a *divisor on* Ω is understood either as a nonnegative integer-valued function Λ with a support $\operatorname{supp}\Lambda=\{\lambda_n\}\subset\Omega$ which is a sequence consisting of pairwise distinct point λ_n and having no limit points in Ω or as a function identically equal to infinity on Ω. With each nonzero function $f\not\equiv 0$ in $H(\Omega)$, the *divisor* Z_f of its *zeros* which is equal at each point $\lambda\in\Omega$ to the multiplicity of the root of f at λ is naturally associated. If $f\equiv 0$ in Ω, then, by definition, $Z_f(\lambda)\equiv\infty$, $\lambda\in\Omega$.

For a submodule I of the module $\mathscr{P}$, we set $Z_I\overset{\text{def}}{=}\min\{Z_f : f\in I\}$ and call Z_I the *divisor of* I.

A space (module) $\mathscr{P}$ is said to be *stable* (with respect to the division by the binomials $z-\lambda$) if the condition $f(z)/(z-\lambda)\in H(\Omega)$ implies that $f(z)/(z-\lambda)\in\mathscr{P}$ for all functions $f\in\mathscr{P}$ and an arbitrary point $\lambda\in\mathbb{C}$.

With each divisor Λ on Ω, we associate the submodule $I(\Lambda)\overset{\text{def}}{=}\{f\in\mathscr{P} : Z_f\geqslant\Lambda\}$ of the module $\mathscr{P}$. It is clear that if $\mathscr{P}$ is a stable module and if $I(\Lambda)\neq\{0\}$ (i.e., $I(\Lambda)$ is a nonzero submodule), then $Z_{I(\Lambda)}=\Lambda$. If $\Lambda=Z_f$ for some function $f\in\mathscr{P}$, then, instead of $I(\Lambda)$, we also write I_f. We also note that, generally speaking, $I_f\neq\bar{I}(f)$, i.e.,I_f is not necessarily a principal submodule.

Q1 A submodule I of the module $\mathscr{P}$ is said to *admit a local description* or to be *abundant** [1–3] if $I=I(Z_I)$.

Let $\mathscr{P}$ be a stable module. Then, by construction, a submodule of the form $I(\Lambda)$ in $\mathscr{P}$ is abundant. In this case, $\bar{I}(f)=I_f$ if and only if the principal submodule $\bar{I}(f)$ is abundant.

Abundant submodules (and ideals) and their properties play an important role in the spectral theory of operators, in questions of approximation, etc. (For more detail, see the introductions in [1–3] and surveys [5, 6]).

A submodule I of the module $\mathscr{P}$ is *externally stable at a point* $\lambda\in\mathbb{C}\setminus\Omega$ if $f(z)/(z-\lambda)\in I$ for each function $f\in I$ (see [1–3]), and I is said to be *externally stable* if it is externally stable at

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*Some authors also use the terms *divisorial, local* [6], *saturated* [15], *determined* (*representable*) *by zeros*, or *synthesized* submodule [5], etc.

each point $\lambda \in \mathbb{C} \setminus \Omega$. A submodule $I \subset \mathscr{P}$ is *internally stable at a point* $\lambda \in \Omega$ if the conditions $f \in I$ and $Z_f(\lambda) > Z_I(\lambda)$ imply that $f(z)/(z-\lambda) \in I$, and I is sad to be *internally stable* if it is internally stable at each point $\lambda \in \Omega$. A submodule $I \subset \mathscr{P}$ is *stable* if it is both externally and internally stable. It is clear that every abundant submodule in a stable module is stable.

A space $\mathscr{P}$ is said to be *uniformly stable* if, for an arbitrary neighborhood of zero $V \subset \mathscr{P}$, there is a neighborhood of zero $U \subset \mathscr{P}$ such that the set $\{f(z)/(z-\lambda) \in H(\Omega) : f \in U,\, \lambda \in \mathbb{C}\}$ is contained in V.

A space $\mathscr{P}$ is said to be *b-stable* (see [1–3]) if, for an arbitrary set B bounded in $\mathscr{P}$, the set $\{f(z)/(z-\lambda) \in H(\Omega) : f \in B,\, \lambda \in \mathbb{C}\}$ is contained in the space $\mathscr{P}$ and is bounded in it.

A uniformly stable space $\mathscr{P}$ is b-stable. Clearly, a uniformly stable or b-stable space is stable. Stability conditions require that the topology of $\mathscr{P}$ be in a sense "soft." In particular, in view of these conditions, the Banach spaces of analytic functions with "rigid" topologies prescribing the existence of more or less explicit boundary values of the functions lie outside the range of application of the result in this paper. At the same time, a wide class of spaces of holomorphic functions is already included in the classes of uniformly stable or b-stable spaces (see Proposition 5.1) if there is even a very slight "gap" between the seminorms determining this topology.

A space (module) $\mathscr{P}$ is said to be *analytically densified* (see [2, Sec. 5]) if, for an arbitrary finite system $f_1, \dots, f_k \in \mathscr{P}$, the set $\{f \in H(\Omega) : |f(z)| \leqslant |f_1(z)| + \dots + |f_k(z)|,\, z \in \Omega\} \subset \mathscr{P}$ is bounded in $\mathscr{P}$.

Let $\ell \in \mathscr{P} \subset H(\Omega)$. If $\ell \not\equiv 0$ and if there is a function $m(z)$, $z \in \Omega$, satisfying the condition

$$\inf_{z \in K} m(z) > 0 \quad \textit{and} \quad \sup_{z \in K} m(z) < +\infty \quad \textit{for an arbitrary compact set } K \subset \Omega \tag{$*$}$$

and possessing the property that the *set*

$$\{f \in \mathscr{P} : |f(z)| \leqslant \max\{|\ell(z)|, m(z)\},\, z \in \Omega\} \tag{1.1}$$

is *bounded in* $\mathscr{P}$, then the function ℓ will be referred to as being *weakly bounding* (in $\mathscr{P}$). If the stronger condition that the *set* Q2

$$\{f \in H(\Omega) : |f(z)| \leqslant \max\{|\ell(z)|, m(z)\},\, z \in \Omega\} \tag{1.2}$$

is *contained in the space* $\mathscr{P}$ *and is bounded in it* holds in this definition ($\mathscr{P}$ is replaced by $H(\Omega)$ in (1.1)), then the function ℓ will be referred to as being *densely weakly bounding* (in $\mathscr{P}$).

The above two definitions impose rather weak constraints on the related objects. For example, if $\mathscr{P}$ is an analytically densified space and if, for a function $\ell \in \mathscr{P}$, there is a function $g \in \mathscr{P}$ having no common zeros with ℓ (in particular, if $\mathscr{P}$ contains a nonvanishing function g), then, setting $m = |\ell| + |g|$ in (1.2), we conclude that ℓ is densely weakly bounding in $\mathscr{P}$.

Let $\ell \in \mathscr{P} \subset H(\Omega)$. If $\ell \not\equiv 0$ and if there is a function m satisfying condition $(*)$ and also a continuous function $\Phi(z) \geqslant 1$, $z \in \mathbb{C}$, *increasing faster than any power function*, i.e.,

$$\log(1 + |z|) = o(\log \Phi(z)), \qquad z \to \infty, \tag{1.3}$$

such that the *set*

$$\{f \in \mathscr{P} : |f(z)| \leqslant \Phi(z) \max\{|\ell(z)|, m(z)\},\, z \in \Omega\} \tag{1.4}$$

is bounded in $\mathscr{P}$, then the function $\ell \not\equiv 0$ will be referred to as being *bounding* (in $\mathscr{P}$). If, instead of (1.4), the stronger condition that the *set*

$$\{f \in H(\Omega) : |f(z)| \leqslant \Phi(z) \max\{|\ell(z)|, m(z)\},\, z \in \Omega\} \tag{1.5}$$

is contained in the space $\mathscr{P}$ *and is bounded in it* holds in this definition ($\mathscr{P}$ is replaced by $H(\Omega)$ in (1.4)), then the function ℓ will be referred to as being *densely bounding* (in $\mathscr{P}$).

For a bounded domain Ω, every (densely) weakly bounding function ℓ in $\mathscr{P}$ is automatically (densely) bounding with an arbitrary continuous function $\Phi \geqslant 1$ in (1.5) or in (1.4). In other words, the appearance of the function Φ in (1.4) or (1.5) is essential only for unbounded domains Ω.

Theorem 1. *Let* $\mathscr{P} \subset H(\Omega)$ *be an analytically densified and sequentially complete module satisfying one of the conditions* [us] *or* [bs] *below.*

[us] $\mathscr{P}$ *is a uniformly stable space,*

[bs] $\mathscr{P}$ *is a bornological* b*-stable space.*

Let an abundant submodule I *in* $\mathscr{P}$ *contain a bounding function* ℓ*. Then it is generated by two submodules* I_ℓ *and* I_g*, where* g *is a function belonging to* I*. In particular, if all principal submodules in* $\mathscr{P}$ *are abundant*[*]*, then* I *is* 2*-generated. More precisely,* $I = \bar{I}(\ell, g)$ *for some function* $g \in I$*.*

The condition of sequential completeness for the module $\mathscr{P}$ *can be removed here if* ℓ *is densely bounding.*

In what follows, the topological closure of a set B will be denoted by $\overline{B}$.

The theorem below deals with a more "individualized" version of the same topic as in Theorem 1.

Theorem 2. *Let* Ω *be either a bounded simply connected domain satisfying the condition*

$$\overline{\mathbb{C} \setminus \overline{\Omega}} = \mathbb{C} \setminus \Omega \tag{1.6}$$

or the space $\mathbb{C}$ *itself,* $\Omega = \mathbb{C}$*, let* $\mathscr{P}$ *be the same module as in Theorem* 1*, and let* I *be a closed internally stable submodule in* $\mathscr{P}$*. If* I *contains an abundant submodule* L *with a bounding function* $\ell \in L$*, then* I *is also abundant and is generated by a pair consisting of the submodule* L *and the principal submodule* $\bar{I}(g)$ *generated by some function* $g \in I$*, i.e.,* $I = \overline{L + \bar{I}(g)}$*.*

Moreover, if, under the same conditions, I *contains a bounding function* ℓ *generating the abundant principal submodule* $\bar{I}(\ell)$*, then the submodule* I *is abundant, and there is a function* $g \in I$ *such that* $I = \bar{I}(\ell, g)$*.*

2. Constructing the Function g from a Given Function ℓ

Given a function ℓ, we suggest a method of constructing the function g from ℓ with a set of analytic properties which is needed in the proof of Theorems 1 and 2 to ensure the required "closeness" between g and ℓ.

We denote by $D(\lambda, t)$ an open disk of radius t with center at $\lambda \in \mathbb{C}$. In particular, if $t \leqslant 0$, then $D(\lambda, t)$ is an empty set.

To represent the divisor $\Lambda \not\equiv \infty$, $\operatorname{supp} \Lambda = \{\lambda_n\}$, $k_n = \Lambda(\lambda_n)$, in Ω we will use the *canonical form* $\Lambda = \{(\lambda_n, k_n)\}$ as well.

Proposition 2.1. *Let* Ω*,* $0 \in \Omega$*, be a domain, let* $\ell \in H(\Omega)$*, and let* $\ell(0) = 1$*. Let a divisor* M *be a subdivisor of the divisor of zeros of* ℓ*, i.e.,* $\mathrm{M} \leqslant Z_\ell$*.*

Then, for an arbitrary function m *satisfying condition* $(*)$ *in Sec.* 1 *and for each continuous function* $\Phi(z) \geqslant 1$*,* $z \in \mathbb{C}$*, increasing faster than any power function in the sense of* (1.3)*, there is a sequence of functions* $\{\ell_n\} \subset H(\Omega)$ *which converges in the topology of uniform convergence to some function* $g \in H(\Omega)$ *such that*

(1) $g(0) = 1$ *and* $\min\{Z_g, Z_\ell\} = \mathrm{M}$,

(2) *the relation*

$$\lim_{k,p\to\infty} \sup_{z\in\Omega} \frac{|\ell_k(z) - \ell_p(z)|}{\max\{|\ell(z)|, m(z)\}} = 0 \tag{2.1}$$

holds,

(3) *the inequality*

$$|g(z)| \leqslant C \max\{\ell(z), m(z)\}, \quad \text{where } z \in \Omega \text{ and } C \text{ is a constant,} \tag{2.2}$$

holds,

(4) *there are two polynomial sequences* $\{p_n\}$ *and* $\{q_n\}$ *satisfying the condition* $\lim_{n\to\infty} p_n(0) = \lim_{n\to\infty} q_n(0) = 1$ *and possessing the property that*

$$\lim_{n\to\infty} \sup_{z\in\Omega} \frac{|p_n(z)\ell(z) - q_n(z)g(z)|}{\Phi(z)\max\{|\ell(z)|, m(z)\}} = 0. \tag{2.3}$$

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[*] Such a module $\mathscr{P}$ is said to be *pure*. (See [2, Sec. 6, Subsec. 1]).

Proof. We consider the divisor $\Lambda = Z_\ell - \mathrm{M}$ and use its canonical representation $\Lambda = \{(\lambda_n, k_n)\}$. The function g will be constructed as the limit of the partial sums[*]

$$\ell_N(z) = \sum_{n=1}^{N} c_n \frac{\ell(z)}{(1-z/\lambda_n)^{k_n}} = \ell(z) \sum_{n=1}^{N} \frac{c_n}{(1-z/\lambda_n)^{k_n}}, \qquad N = 1, 2, \dots . \tag{2.4}$$

Let us concretize the construction of the sequence $\{c_n\}$. For this, we take a sequence of *pairwise disjoint disks* $D(\lambda_n, \varepsilon_n)$, $\varepsilon_n > 0$, $n = 1, 2, \dots$, such that

$$|\ell(z)| \leqslant \inf_{\zeta \in D(\lambda_n, \varepsilon_n)} m(\zeta), \qquad z \in D(\lambda_n, \varepsilon_n),\ n = 1, 2, \dots . \tag{2.5}$$

This can always be done consecutively for $n = 1, 2, \dots$ according to the maximum principle for $|\ell|$ and in view of condition $(*)$ on the functions m since $\ell(\lambda_n) = 0$.

We now select a sequence$\{c_n\}$ satisfying the following conditions:

(c_0) $c_n > 0$, $n = 1, 2, \dots$, and $\sum_{n \geqslant 1} c_n = 1$,

(c_*) the estimate

$$a_N \sum_{n \geqslant N+1} \frac{c_n |\lambda|^{k_n}}{\varepsilon_n^{k_n}} = o(1) \quad \text{as } N \to \infty \tag{2.6}$$

holds, where

$$a_N = \begin{cases} \displaystyle\prod_{n=1}^{N} \Big(1 + \frac{\sup_{z\in\Omega} |z|}{|\lambda_n|}\Big)^{k_n} & \text{if } \Omega \text{ is bounded,} \\ \displaystyle\sup_{z\in\Omega} \frac{\prod_{n=1}^{N} \Big(1 + \frac{|z|}{|\lambda_n|}\Big)^{k_n}}{\Phi(z)} & \text{if } \Omega \text{ is unbounded.} \end{cases} \tag{2.7}$$

If the domain Ω is bounded, then the coefficients $a_N > 0$ in (2.7) are obviously finite, and if it is unbounded, then this is true since Φ increases faster than any power function. The required choice of the sequence $\{c_n\}$ can also be realized on the basis of the elementary lemma below.

Lemma 2.1. *Let* $\{a_n\}$ *and* $\{b_n\}$, $n = 1, 2, \dots$, *be sequences of positive numbers. Then there is a sequence* $\{c_n\}$ *such that condition* (c_0) *is fulfilled and the relation* $a_N \sum_{n \geqslant N+1} c_n b_n = o(1)$ *as* $N \to \infty$ *holds.*

Proof. We set $s_1 = 1$ and $s_n = 1/a_{n-1} > 0$, $n = 2, 3, \dots$. By construction, the sequence $\{s'_n\}$, $s'_1 = 1$, $s'_n = \min\{s_1, \dots, s_n\}/(n-1)$, $n \geqslant 2$, is strictly monotone decreasing. We write $c'_n = \min\{(s'_n - s'_{n+1})/b_n, 1/n^2\} > 0$. By construction, the series $\sum_{n\geqslant 1} c'_n \leqslant \sum_{n \geqslant 1} 1/n^2$ converges to a number $S > 0$ in such a way that the relation $\sum_{n \geqslant N+1} c'_n b_n \leqslant \sum_{n \geqslant N+1} (s'_n - s'_{n+1}) = s'_{N+1} \leqslant s_{N+1}/N = 1/(N a_N)$ holds. Setting $c_n = c'_n/S$, we obtain the desired sequence $\{c_n\}$.

Applying Lemma 2.1 for a_N in (2.7) and for $b_n = |\lambda|^{k_n}/\varepsilon_n^{k_n}$, we obtain a sequence $\{c_n\}$ with properties (c_0) and (c_*). We thus derive an upper estimate for the moduli of the terms in (2.4). For $z \notin D(\lambda_n, \varepsilon_n)$, we have

$$\left| \frac{\ell(z)}{(1-z/\lambda_n)^{k_n}} \right| \leqslant |\ell(z)| \frac{|\lambda_n|^{k_n}}{\varepsilon_n^{k_n}}, \qquad z \notin D(\lambda_n, \varepsilon_n). \tag{2.8}$$

For $z \in D(\lambda_n, \varepsilon_n)$, by the maximum principle in (2.5), it follows that

$$\left| \frac{\ell(z)}{(1-z/\lambda_n)^{k_n}} \right| \leqslant m(z) \frac{|\lambda_n|^{k_n}}{\varepsilon_n^{k_n}}, \qquad z \in D(\lambda_n, \varepsilon_n). \tag{2.9}$$

Hence, for all $z \in \Omega$, formulas (2.8) and (2.9) imply that

$$\left| c_n \frac{\ell(z)}{(1-z/\lambda_n)^{k_n}} \right| \leqslant \max\{|\ell(z)|, m(z)\}\, c_n \frac{|\lambda_n|^{k_n}}{\varepsilon_n^{k_n}}, \qquad z \in \Omega, \tag{2.10}$$

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[*] As usual, the sum (product) of an empty set of elements is understood as being equal to zero (unity).

whence, for $1 \leqslant p \leqslant k$, by the definition of the functions ℓ_N in (2.4), it follows that

$$|\ell_k(z) - \ell_p(z)| \leqslant \max\{|\ell(z)|, m(z)\} \sum_{n=p+1}^{k} c_n \frac{|\lambda_n|^{k_n}}{\varepsilon_n^{k_n}}, \qquad z \in \Omega. \tag{2.11}$$

By condition (c_*) for the choice of the sequence $\{c_n\}$, where the sequence $\{a_N\}$ defined by formula (2.7) is nondecreasing, the series on the left-hand side of (2.6) is convergent. Consequently, (2.11) implies (2.1), and the sequence $\{\ell_N\}$ in (2.4) is uniformly convergent (together with its all derivatives) on the compact sets to the function

$$g(z) = \sum_{n=1}^{\infty} c_n \frac{\ell(z)}{(1 - z/\lambda_n)^{k_n}} \in H(\Omega) \tag{2.12}$$

satisfying estimate (2.2). Furthermore, by definition (2.4) of the functions ℓ_N and in view of condition (c_0), we have $g(0) = \sum_{n \geqslant 1} c_n = 1$. Thus, properties (2) and (3) in the proposition hold, and, to prove that (1) is fulfilled, it remains to show that $\min\{Z_g, Z_\ell\} = \mathrm{M}$.

If $\lambda \notin \operatorname{supp} \Lambda$, i.e., $\Lambda(\lambda) = 0$, then each of the terms in (2.12) has a root of multiplicity $Z_\ell(\lambda)$ (possibly of multiplicity 0) at the point λ. Consequently, the function g has a root of multiplicity no lower than $Z_\ell(\lambda)$ at λ, and the relation

$$\min\{Z_\ell(\lambda), Z_g(\lambda)\} = Z_\ell(\lambda) = Z_\ell(\lambda) - \Lambda(\lambda) = \mathrm{M}(\lambda), \qquad \lambda \notin \operatorname{supp} \Lambda, \tag{2.13}$$

holds. If $\lambda = \lambda_n \in \operatorname{supp} \Lambda$, i.e., $\Lambda(\lambda) > 0$, then each of the terms in (2.12) except the nth summand has a root of multiplicity $Z_\ell(\lambda_n)$ at the point λ_n, and this nth summand has a root of multiplicity $Z_\ell(\lambda_n) - k_n = Z_\ell(\lambda_n) - \Lambda(\lambda_n) = \mathrm{M}(\lambda_n) < Z_\ell(\lambda_n)$ at λ_n. Thus, the function g has a root of multiplicity $Z_g(\lambda) = \mathrm{M}(\lambda) < Z_\ell(\lambda)$ at each of the points $\lambda \in \operatorname{supp} \Lambda$, and the relation $\min\{Z_\ell(\lambda), Z_g(\lambda)\} = Z_g(\lambda) = \mathrm{M}(\lambda)$, $\lambda \in \operatorname{supp} \Lambda$, holds. This, together with (2.13), implies that $\min\{Z_g, Z_\ell\} = \mathrm{M}$, i.e., property (1) in the proposition holds.

It remains to establish property (4), i.e., relation (2.3). We set

$$q_N(z) = \prod_{n=1}^{N} \left(1 - \frac{z}{\lambda_n}\right)^{k_n} \tag{2.14}$$

and, with regard to representations (2.12) and (2.4), consider the identity

$$q_N(z) g(z) = \ell(z) p_N(z) + q_N(z) R_N(z), \tag{2.15}$$

where

$$p_N(z) = \sum_{n=1}^{N} \frac{c_n q_N(z)}{(1 - z/\lambda_n)^{k_n}}, \qquad R_N(z) = \sum_{n=N+1}^{\infty} \frac{c_n \ell(z)}{(1 - z/\lambda_n)^{k_n}}\,. \tag{2.16}$$

Here, by the construction in (2.14), the function p_N is a *polynomial*, and we have

$$p_N(0) = \sum_{n=1}^{N} c_n \to 1 \quad \text{as } N \to \infty. \tag{2.17}$$

According to (2.10), the remainder R_N of the series satisfies the inequality

$$|R_N(z)| \leqslant \max\{|\ell(z)|, m(z)\} \sum_{n=N+1}^{\infty} c_n \frac{|\lambda_n|^{k_n}}{\varepsilon_n^{k_n}}, \qquad N = 1, 2, \dots\,. \tag{2.18}$$

Furthermore, definition (2.14) implies the inequality

$$|q_N(z)| \leqslant \prod_{n=1}^{N} \left(1 + \frac{|z|}{|\lambda_n|}\right)^{k_n}, \qquad q_N(0) = 1,\ N = 1, 2, \dots\,. \tag{2.19}$$

By estimates (2.18) and (2.19), it follows from representation (2.15)–(2.16) that

$$\begin{aligned}|p_N(z)\ell(z) - q_N(z)g(z)| &\leqslant |q_N(z)R_N(z)| \\ &\leqslant \prod_{n=1}^{N}\left(1+\frac{|z|}{|\lambda_n|}\right)^{k_n} \max\{|\ell(z)|, m(z)\} \sum_{n=N+1}^{\infty} c_n \,\frac{|\lambda_n|^{k_n}}{\varepsilon_n^{k_n}}\,. \end{aligned} \tag{2.20}$$

If Ω is a bounded domain, then we rewrite (2.20) using notation (2.7) and then, by relation (2.6), derive (2.3) from condition (c_*) on the sequence$\{c_n\}$ since $\Phi(z) \geqslant 1$. If Ω is an unbounded domain, then, preliminarily dividing and multiplying the right-hand side of (2.20) by $\Phi(z)$, we again derive (2.3) from (2.20) using the same procedure as above. Thus, the proof is complete.

Remark. Along with the additive method of constructing the function g as the limit of the partial sums ℓ_N in (2.4), an alternative multiplicative method is also possible (cf. [11–13, 16]) in which g is constructed as the limit of a recurrent sequence $\ell_0 = \ell$, $\ell_n(z) = (1-z/\gamma_n)^{k_n}\ell_{n-1}(z)/(1-z/\lambda_n)^{k_n} \in H(\Omega)$, $n = 1, 2, \dots$, where $\{\gamma_n\}$ is sufficiently close to the sequence $\{\lambda_n\}$, but has no common points with it. This method leads to the same result, but, in our realization, it turned out to be much more extensive and laborious.

Proposition 2.2. *Let* $0 \in \Omega$, *let* $\mathscr{P} \subset H(\Omega)$ *be a stable sequentially complete space, and let* $\ell \in \mathscr{P}$ *be a weakly bounding function. Then the function* g *constructed as in Proposition* 2.1 *belongs to* $\mathscr{P}$. *The conditions of sequential completeness and stability for the space* $\mathscr{P}$ *can be removed here if* ℓ *is a densely weakly bounding function.*

Proof. If g is a *densely* weakly bounding function in $\mathscr{P}$, then, according to estimate (2.2) for g, it belongs to the set (1.2), which is contained in $\mathscr{P}$. Hence, $g \in \mathscr{P}$, which proves the second part of the propositions.

If ℓ is only a weakly bounding function belonging to the stable sequentially complete space $\mathscr{P}$, then, by the stability of this space, all the function ℓ_N constructed by rule (2.4) belong to $\mathscr{P}$. Since the set (1.1) is bounded, relation (2.1) in Proposition 2.1 means that $\{\ell_n\}$ is a Cauchy sequence. Consequently, its limit in $\mathscr{P}$, which coincides with g since the topology of the space $\mathscr{P}$ majorizes the topology of pointwise convergence, also belongs to $\mathscr{P}$.

3. On Submodules Generated by Two Submodules

Throughout this section, Ω is a domain in $\mathbb{C}$, $0 \in \Omega$, and it is assumed that the module $\mathscr{P}$ satisfies either condition [us] or condition [bs]. (See Theorem 1.) By Remark 2 in [2, Sec. 4, Subsec. 2], the condition of *pointwise stability* holds in the case [bs] and, obviously, in the case [us] as well. Thus, the assertion below is true.

Let V *be an arbitrary neighborhood of zero in* $\mathscr{P}$. *Then, for each* $\lambda \in \mathbb{C}$, *there is a neighborhood* U_λ *such that the conditions* $f \in U_\lambda$ *and* $f(z)/(z-\lambda) \in H(\Omega)$ *imply that* $f(z)/(z-\lambda) \in V$.

Preliminarily, we present a list of some more results in [2] needed in what follows and give some comments when necessary.

(i) *If a net* (*generalized sequence*) $\{f_\sigma\} \subset \mathscr{P}$ *over a directed set* $\{\sigma\}$ *converges in* $\mathscr{P}$ *to an element* $f \in \mathscr{P}$, *then, for each* $n = 0, 1, \dots$, *the* n*th derivatives satisfy the condition* $f_\sigma^{(n)}(\lambda) \underset{\sigma}{\to} f^{(n)}(\lambda)$ *at all points* $\lambda \in \Omega$. (See [2, Proposition 4.5 and Sec. 4, Subsec. 2 and Remark 2].) This property ensures the preservation of the divisor of a submodule under the transition to its closure, and we will use it in what follows without any additional reference.

(ii) *A closed submodule in* $\mathscr{P}$ *internally stable at a point in* Ω *is internally stable.* (See [2, Proposition 4.2 and Sec. 4, Remark 1].) Q3

(iii) *The closure* $\bar{I}$ *of a submodule* $I \subset \mathscr{P}$ *stable at a point* λ *is stable at the same point* (see [2, Proposition 4.5 and Sec. 4, Subsec. 2, Remark 2]).

(iv) *In an analytically densified module* $\mathscr{P}$, *every closed stable submodule containing an abundant nonzero submodule is also abundant* [2] (see Proposition 6.4). This result was stated in [2] for submodules over $\mathbb{C}[z]$ of local rank 1 in modules of vector functions. But in the case of scalar

Q4 functions, all this is just as well true without the above additional assumption since every subset in $H(\Omega)$ distinct from $\{0\}$ is of local rank 1. (See [2, Sec. 1, Subsec. 3].) Moreover, this result was stated in [2] only for the case [us] without any comment on the case [bs]. By the assertion in the very end of [2, Sec. 5], it is readily seen that the indicated result is true in the case [bs] as well.

The two propositions below are a slightly different version of essentially more general assertions in [2] (namely, of Propositions 4.8 and 6.6). Cf. [17, Proposition 5.6].

Proposition 3.1. *Let J be a closed submodule in $\mathscr{P} \subset H(\Omega)$ generated by a pair of submodules L and G, let $\ell \in L$ and $g \in G$, $\ell(0) = g(0) = 1$, and let the following assumption hold*:

[ss] *there are two polynomial sequences $\{p_n\}, \{q_n\} \in \mathbb{C}[z]$, $n = 1, 2, \dots$, satisfying the condition*

$$\lim_{n\to\infty} p_n(0) = \lim_{n\to\infty} q_n(0) = 1 \tag{3.1}$$

and possessing the property that the sequence $\{p_n\ell - q_n g\}$, $n = 1, 2, \dots$, tends to $0 \in \mathscr{P}$ in the topology of the space $\mathscr{P}$ as $n \to \infty$.

If the submodules L and G are internally stable at the point $0 \in \Omega$, then the submodule J is also internally stable.

Proof. We first somewhat modify the sequences $\{p_n\}$ and $\{q_n\}$, $n = 1, 2, \dots$. By the definition of a topological vector space, it follows from condition (3.1) that $(p_n(0) - 1)\ell - (q_n(0) - 1)g \underset{n}{\to} 0$ in $\mathscr{P}$. What has been established implies that the sequence $\{(p_n - (p_n(0) - 1))\ell - (q_n - (q_n(0) - 1))g\}$ converges to 0 in $\mathscr{P}$, and the polynomials for ℓ and g are already equal to 1 at zero. Consequently, condition (3.1) in the proposition can be replaced by

$$p_n(0) = q_n(0) = \ell(0) = g(0) = 1, \quad n = 1, 2, \dots, \qquad p_n\ell - q_n g \underset{n}{\to} 0 \text{ in } \mathscr{P}. \tag{3.2}$$

It is clear that $Z_J(0) = 0$ since $\ell(0) = 1 \neq 0$ and $\ell \in L \subset J$. We consider an arbitrary element $F \in J$ such that $Z_F(0) > Z_J(0) = 0$. By (ii), it suffices to show that $F(z)/z \in J$.

Let V be a neighborhood of zero in $\mathscr{P}$. It follows from the remark at the beginning of this section that, in each of the cases [us] and [bs], the module $\mathscr{P}$ is pointwise stable. Hence, there is a neighborhood of zero U_0 in $\mathscr{P}$ such that the implication

$$f \in U_0,\ f(0) = 0 \implies \frac{f(z)}{z} \in V \tag{3.3}$$

is true. We now choose a neighborhood of zero W such that $W + W \subset U_0$. Since $F \in J$, there is a net $F_\sigma = \ell_\sigma - g_\sigma \in L + G$ over the directed set $\{\sigma\}$, $\ell_\sigma \in L$, $g_\sigma \in G$, convergent to F in $\mathscr{P}$. We modify it in such a way that the terms of the net vanish at the point $0 \in \Omega$. For this, it suffices to consider another net $F^*_\sigma = F_\sigma - (\ell_\sigma(0) - g_\sigma(0))\ell = \ell_\sigma - g_\sigma - (\ell_\sigma(0) - g_\sigma(0))\ell \in L + G$, which, as before, converges to F in $\mathscr{P}$ since $F_\sigma(0) \to F(0) = 0$, $\ell(0) = 1$, and, accordingly, $(\ell_\sigma(0) - g_\sigma(0))\ell \to 0$ in $\mathscr{P}$. Here, the new net contains a term having the form $F^* = f_L - f_G \in L + G$, where $f_L = \ell_\sigma - (\ell_\sigma(0) - g_\sigma(0))\ell \in L$, $f_G = g_\sigma \in G$, and $F^*(0) = f_L(0) - f_G(0) = 0$, and possessing the property that $F - F^* \in W$. Let us set $h_n = f_L(0)(p_n\ell - q_n g) = f_L(0)p_n\ell - f_G(0)q_n g \in W$ for a sufficiently large n. For this value of n, we have $F - F^* + h_n \in W + W \subset U_0$, and, in view of (3.3), the division by z now gives

$$\frac{F(z)}{z} - \frac{F^*(z) - h_n(z)}{z} = \frac{F(z)}{z} - \left(\frac{f_L - f_L(0)p_n\ell}{z} - \frac{f_G - f_G(0)q_n g}{z}\right) \in V,$$

where, with regard to (3.2) and by the internal stability of L and G at $0 \in \Omega$, the inclusion relations $(f_L - f_L(0)p_n\ell)/z \in L$ and $(f_G - f_G(0)q_n g)/z \in G$ hold.

By the arbitrariness in the choice of V, these relations mean that $F(z)/z$ is a point of contact for the sum $L + G$, i.e., $F(z)/z \in J$, which is what we had to prove.

Proposition 3.2. *Let, in addition to property* [us] *or* [bs], *the module $\mathscr{P}$ be analytically densified and let the assumptions of Proposition* 3.1 *including condition* [ss] *also hold. If the submodules L and G are abundant, then the submodule $J = \overline{L + G}$ generated by them is also abundant.*

Proof. As was noted in Sec. 1, by the definition of the property of abundance, each of the submodules L and G is stable. By definition, J is a closed submodule. According to Proposition 3.1, it is internally stable. Since each of the submodules L and G is externally stable, their sum $L+G$ is also externally stable. (This sum is regarded here without closure. See the definition of external stability.) In the topological module, the closure $J = \overline{L+G}$ of the submodule $L+G$ is also a submodule, and, according to (iii), the submodule J is also externally stable. Thus, J is closed and both internally and externally stable, i.e., it is a stable submodule. By construction, J contains the abundant submodule L (or G), whence, since the module $\mathscr{P}$ is analytically densified, it follows that, according to (iv), the submodule J is abundant, which is what is required to prove.

4. Proofs of Theorems 1 and 2

Using the shift of the complex plane, it can always be assumed without loss of generality in the proofs of Theorems 1 and 2 that $0 \in \Omega$ and $\ell(0) = 1$. Throughout this section, it will also be assumed that the module $\mathscr{P}$ satisfies one of the conditions [us] and [bs] in the statement of Theorem 1.

The two facts presented below continue the list of the results in [2] (see the previous section) we need.

(v) A *principal submodule in $\mathscr{P}$ is always internally stable.* (See [2, Sec. 4, Susbec. 3, Corollary 2].) This assertion was proved only in the case [us], but an analysis of the proof of Proposition 4.7 in [2], whose corollary has just been stated, shows that only the property of b-stability and the property of pointwise stability, which follows from [bs], are in fact used in this proposition.

(vi) *For a bounded simply connected domain Ω satisfying condition* (1.6)*, every closed submodule $I \subset \mathscr{P}$ is externally stable.* (See [2, Proposition 4.4 and Sec. 4, Susbec. 1, Remark 1].)

Proposition 4.1. *Let $\mathscr{P}$ be the same module as in Theorem* 1*, let ℓ be a bounding function in $\mathscr{P}$, and let M be a divisor such that $\mathrm{M} \leqslant Z_\ell$. Then there is a function $g \in \mathscr{P}$ such that the submodule J generated by the submodules I_ℓ and I_g is abundant and the relation $Z_J = \mathrm{M}$ holds. The condition of sequential completeness for the module $\mathscr{P}$ can be removed if ℓ is a densely bounding function.*

Proof. We choose g in the form of a function constructed as in Proposition 2.1. Proposition 2.2 Q5
implies that $g \in \mathscr{P}$.

Both the submodules I_ℓ and I_g are abundant since they have the form $I(\Lambda)$. (See Sec. 1.) By the boundedness of the set (1.4) (of the set (1.5) if ℓ is densely bounding), relation (2.3) in property (4) stated in Proposition 2.1 means that condition [ss] in Proposition 3.1 holds. By Proposition 3.2, it follows that the submodule $J = \overline{I_\ell + I_g}$ is abundant. In this case, by the construction of the submodules J, I_ℓ, and I_g, we have $Z_J = \min\{Z_\ell, Z_g\}$. According to property (1) in Proposition 2.1, the relation $\min\{Z_\ell, Z_g\} = \mathrm{M}$ holds, and hence $Z_J = \mathrm{M}$, which proves the proposition.

Proof of Theorem 1. Let $\mathrm{M} = Z_I$. It is clear that $\mathrm{M} \leqslant Z_\ell$. By Proposition 4.1, the submodule $J = \overline{I_\ell + I_g}$ is abundant as well as the submodule I, and we have $Z_J = \mathrm{M} = Z_I$. Two abundant submodules with the same divisors coincide, i.e., $I = J$. If all principal submodules in $\mathscr{P}$ are abundant, then (see Sec. 1) $I_\ell = \bar{I}(\ell)$ and $I_g = \bar{I}(\ell)$, i.e., $J = \overline{\bar{I}(\ell) + \bar{I}(g)}$, which proves the theorem

Proposition 4.2. *Let the module $\mathscr{P}$, the domain Ω, and the function ℓ be respectively the same* Q6
as in Theorem 1*, Theorem* 2*, and Proposition* 4.1*. Let L be an abundant submodule in $\mathscr{P}$ which contains ℓ and let M be a divisor such that $\mathrm{M} \leqslant Z_L$. Then there is a function $g \in \mathscr{P}$ such that the submodule J generated by the submodule L and by the principal submodule $\bar{I}(g)$ is abundant and the relation $Z_J = \mathrm{M}$ holds.*

Proof. The condition $\mathrm{M} \leqslant Z_L$ implies that $\mathrm{M} \leqslant Z_\ell$. We again choose g in the form of a Q5
function constructed as in Proposition 2.1, which, by Proposition 2.2, belongs to $\mathscr{P}$. According to (v), the principal submodule $\bar{I}(g)$ is internally stable. The submodule L is stable since it is abundant. Since the set (1.4) (the set (1.5) if ℓ is densely bounding) is bounded, relation (2.3) in

assertion (4) of Proposition 2.1 means that condition [ss] in Proposition 3.1 holds. Consequently, by Proposition 3.1, the submodule J generated by L and $\bar{I}(g)$ is internally stable. It follows from (vi) that the closed submodule J is externally stable if the domain Ω satisfies condition (1.6), i.e., J is stable in this case. Since J contains the abundant submodule L, it follows that, according to (iv), J is abundant .

Furthermore, by property (1) of the function g in Proposition 2.1, we have $\min\{Z_\ell, Z_g\} = \mathrm{M}$, whence

$$Z_J = \min\{Z_L, Z_g\} \leqslant \min\{Z_\ell, Z_g\} = \mathrm{M}. \tag{4.4}$$

By the condition $\mathrm{M} \leqslant Z_L$ and according to the construction, we have $\mathrm{M} \leqslant Z_g$, i.e., $\mathrm{M} \leqslant \min\{Z_L, Z_g\}$, which, together with (4.4), gives $\mathrm{M} = Z_J$, and the proposition is thus proved.

Proof of Theorem 2. We take the divisor $\mathrm{M} = Z_I$. The inclusion relation $L \subset I$ implies that $\mathrm{M} \leqslant Z_L$. The submodule $J = \overline{L + I_g}$ in Proposition 4.2 is abundant.

By assumption, the submodule I is closed and internally stable, and, according to (vi), it is externally stable as well. Therefore, I is a closed stable submodule. Here I includes an abundant submodule L, whence, by (iv), it follows that I is also an abundant submodule. Moreover, we have $Z_J = \mathrm{M} = Z_I$. Consequently, the two abundant submodules I and J coincide, which proves the first part of the theorem.

To deduce the second part from the above, it suffices to note that the abundant principal submodule $\bar{I}(\ell)$ can be taken as the abundant submodule $L \subset I$ entering the assumptions of the theorem.

5. Applications

We first single out a rather wide class of spaces to which Theorems 1 and 2 are applicable.

Let q be a real-valued function in Ω. We denote by $\mathscr{P}_q$ the normed space consisting of all functions $f \in H(\Omega)$ with finite norm $\|f\|_q = \sup_{\zeta\in\Omega}(|f(\zeta)| \exp(-q(\zeta)))$.

Let $Q = \{q_n\}$, $n \in \mathbb{N}$, denote an increasing (decreasing) sequence of real-valued functions on Ω, and let us introduce the locally convex space $\mathscr{P}_Q^{\uparrow} = \bigcup_{n\geqslant 1} \mathscr{P}_{q_n}$ (the space $\mathscr{P}_Q^{\downarrow} = \bigcap_{n\geqslant 1} \mathscr{P}_{q_n}$) with the natural topology of inductive (projective) limit. (See [4, 18].) We also introduce the function $l_\Omega(z) = \max\{1+|z|, 1/\operatorname{dist}(z, \partial\Omega)\}$, $z \in \Omega$, where $\operatorname{dist}(\,\cdot\,,\,\cdot\,)$ is the distance function and $\partial\Omega$ is the boundary of Ω.

Q7 Recall (see [18]) that the inductive limit $\mathscr{P}_Q^{\uparrow}$ (projective limit $\mathscr{P}_Q^{\downarrow}$) is an $(LN)^*$ $((M)^*)$ type space on condition that, for each index n, there is an index $k > n$ (for each k there is an index $n > k$) such that the embedding of $\mathscr{P}_{q_n}$ in $\mathscr{P}_{q_k}$ is completely continuous, i.e., it transforms a ball in the space $\mathscr{P}_{q_n}$ into a relatively compact subset in $\mathscr{P}_{q_k}$. The spaces of the type $(LN)^*$ $((M)^*)$ possess some useful properties essentially facilitating the operation with them. (See [18].)

Proposition 5.1 (see [16, Theorem 2])**.** *Let an increasing* (*decreasig*) *sequence* $Q = \{q_n\}$ *of continuous functions in* $\mathbb{C}$ *satisfy the following conditions*:

[Qs] *for an arbitrary index* n (m), *there are indices* m *and* k *satisfying the inequalities* $m \geqslant k \geqslant n$ (*indices* n *and* k *satisfying the inequalities* $n \geqslant k \geqslant m$) *and a constant* $C \geqslant 0$ *such that, for each point* $\lambda \in \Omega$, *the inequality* $\sup_{\zeta\in D} q_k(\zeta) \leqslant \inf_{\zeta\in D} q_m(\zeta) + C$ *holds in the domain*

$$D = D_{n,k}(\lambda) = \left\{\zeta \in \Omega : |\zeta - \lambda| < \frac{\exp(q_n(\zeta) - q_k(\zeta))}{3\, l_\Omega(\zeta)}\right\},$$

[Ql] *for an arbitrary index* m (p), *there is an index* $p \geqslant m$ ($m \geqslant p$) *and a constant* $C' \geqslant 0$ *such that* $q_m(z) + \log l_\Omega(z) \leqslant q_p(z) + C'$, $z \in \Omega$.

Then the inductive (*projective*) *limit* $\mathscr{P}_Q^{\uparrow}$ ($\mathscr{P}_Q^{\downarrow}$) *is a bornological b-stable* (*uniformly stable*)
Q7 *separable complete reflexive analytically densified space of type* $(LN)^*$ $((M)^*)$ *and, simultaneously,*
a topological module over the ring $\mathbb{C}[z]$.

For a 2π-periodic ρ-trigonometrically convex (see[19]) lower semicontinuous function $H(\theta) > 0$, $\theta \in \mathbb{R}$, the module of entire functions $[\rho, H)$ for $\rho > 0$ ($[\rho, H]$ for $\rho > 1$) with indicator, for the order ρ, strictly less (no less) than $H(\theta)$, $\theta \in [0, 2\pi)$, is a space of the form $\mathscr{P}_Q^{\uparrow}$ ($\mathscr{P}_Q^{\downarrow}$) dealt with in Proposition 5.1. Consequently, Theorem 1 and 2 can be used in these spaces. Following this approach and using the well-known duality scheme when necessary, it is easy to obtain some of the main results in [11, Theorems 1 and 2], [12, Theorem 3], and [13, Theorems 3.6, 4.6, and 4.7] on 2-generated submodules and on the representability of differentiation-invariant subspaces in the form of a solution space of two homogeneous convolution equations. To demonstrate another application of the main results to spectral synthesis, we resort to the classical space $C^\infty(a, b)$ of infinitely differentiable complex-valued functions on an open (bounded or unbounded) interval $(a, b) \subset \mathbb{R}$, $-\infty \leqslant a < b \leqslant +\infty$. (In what follows, some facts given in [4, 20, 21] are used.)

Let $K_n = [a_n, b_n] \subset [a_{n+1}, b_{n+1}]$, $n = 0, 1, \dots$, be closed intervals in $\mathbb{R}$ and let their union coincide with (a, b). We denote by $C^n(K_n)$ the space of functions continuously differentiable up to the order n on $[a_n, b_n]$ with the norm $\|f\|_n = \sup_{x \in K_n} \max_{0 \leqslant k \leqslant n} |f^{(k)}(x)|$, $f \in C^n(K_n)$. The natural topology on the space $C^\infty(a, b) = \bigcap_{n=0}^\infty C^n(K_n)$ is introduced as the projective-limit topology of the space $C^n(K_n)$.

Following L. Schwartz, we denote the strong dual of $C^\infty(a, b)$, which is the space of distributions compactly supported in the interval (a, b), by $\mathscr{E}'(a, b)$ since Schwartz originally denoted the space $C^\infty(a, b)$ by the symbol $\mathscr{E}(a, b)$. The space $\hat{\mathscr{E}}'(a, b)$ consists of the Fourier–Laplace transforms $\hat{S}(z) = \langle S, \exp(-izx)\rangle$, $z \in \mathbb{C}$, of all distributions $S \in \mathscr{E}'(a, b)$, and its topology is induced by that of $\mathscr{E}'(a, b)$ under the Fourier–Laplace transformation. The correspondence determined by this transformation is one-to-one. For $J \subset \mathscr{E}'(a, b)$, we set $\hat{J} = \{\hat{S} : S \in J\}$. If $l \in \hat{\mathscr{E}}'(a, b)$, then $\check{l}$ will denote the compactly supported distribution whose Fourier–Laplace transform is l.

Furthermore, we introduce the spaces

$$\mathscr{P}_n = \left\{ l \in H(\mathbb{C}) : \|l\|_n' = \sup_{z \in \mathbb{C}} \frac{|l(z)|}{(1 + |z|)^n \exp(b_n \operatorname{Im}^+ z - a_n \operatorname{Im}^- z)} < \infty \right\},$$

where $\operatorname{Im}^{\pm}(z) = \max\{0, \pm \operatorname{Im} z\}$. By one of the Paley–Wiener-Schwartz theorems, the space $\hat{\mathscr{E}}'(a, b)$ can be internally described as the inductive limit of the normed spaces $\mathscr{P}_n$, $n = 1, 2, \dots$, with the norms $\| \cdot \|_n'$.

It can be seen easily that $\hat{\mathscr{E}}'(a, b)$ is a space of the form $\mathscr{P}_Q^{\uparrow}$ satisfying the assumptions of Proposition 5.1 and corresponding assertion of the proposition holds for it.

Theorem 3. *Every abundant submodule I in the module $\hat{\mathscr{E}}'(a, b)$ is generated by a pair of submodules of the form I_ℓ and I_g for some $\ell, g \in I$.*

Proof. By Theorem 1, it suffices to show that I contains a bounding function ℓ. Without loss of generality, we can consider the space $\hat{\mathscr{E}}'(-a, a)$ with $a > 0$.

First, let $f \in I$. Then there are numbers $k \in \mathbb{N}$, a, and a', $0 < a' < a$, such that $|f(z)| \leqslant C(1 + |z|)^k \exp(a' |\operatorname{Im} z|)$, $z \in \mathbb{C}$, where C is a constant. We take a number $a'' > 0$ such that $a' < a'' < a$ and an exponential-type entire function $h \not\equiv 0$ satisfying, for instance, an estimate of the form $|h(z)| \leqslant \exp(\varepsilon |\operatorname{Im} z| - \sqrt{|z|})$, where $0 < \varepsilon < a - a''$. (Such a function always exists. For example, see [22, Theorems 39 and 40].) In view of the estimates for f and h, the product $\ell = fh$ belongs to $\mathscr{P}_n \subset \hat{\mathscr{E}}'(-a, a)$ for some $n \geqslant k$ such that $a'' + \varepsilon < a_n$. Moreover, since I is abundant, we have $\ell \in I$ in view of the relation $Z_I \leqslant Z_f \leqslant Z_\ell$. If $m(z) \equiv 1$ and $\Phi(z) = \exp \sqrt{|z|}$, then the set (1.5) is contained in the space $\mathscr{P}_n$ and is bounded in it. For $\hat{\mathscr{E}}'(-a, a)$ regarded as an $(LN)^*$ type space, this means (see [18, Theorem 2]) that the set (1.5) is bounded in $\hat{\mathscr{E}}'(-a, a)$. Hence, ℓ is Q7
a bounding function in I, which is what we had to prove.

For $S \in \mathscr{E}'(a, b)$ on the interval $\{h \in \mathbb{R} : \operatorname{co supp} S + h \subset (a, b)\}$, where $\operatorname{co supp} S$ is the convex hull of the set $\operatorname{supp} S$, the convolution $(S * f)(h) = \langle S, f(x + h)\rangle$ is defined for all $f \in C^\infty(a, b)$.

The (differentiation) invariant subspace $W \subset C^n(a, b)$ *admits spectral synthesis* if it coincides with the closure of the linear span of the functions of the form $x^m e^{i\lambda x} \in W$ in the variable $x \in (a, b)$. The assertion below is established following the standard scheme. (Compare with [21, Sec. 4] and [24, Sec. 2].)

Duality principle. *There is a one-to-one correspondence between the set of all closed invariant subspaces* $W \subset C^\infty(a,b)$ *and the set of all closed submodules* $I \subset \hat{\mathscr{E}}'(a,b)$ *which is established according to the following orthogonality rule:* $W \leftrightarrow I$ *if and only if* $I = \hat{J}$*, where the functionals in* $J \subset \mathscr{E}'(a,b)$ *and only they annihilate* W*. An invariant subspace* W *admits spectral synthesis if and only if the corresponding closed submodule* $I \subset \hat{\mathscr{E}}'(a,b)$ *is abundant.*

Following [17, Sec. 6], we refer to the closed submodule $I \in \hat{\mathscr{E}}'(a,b)$ corresponding to a closed invariant subspace $W \in C^\infty(a,b)$ in accordance with the duality principle as the *annihilator submodule for* W. The annihilator submodule for the intersection $\bigcap_{k=1}^{n} W_k$ coincides with the closed submodule generated by the annihilator submodules $I_1, \dots, I_n$ for the corresponding closed invariant subspaces $W_1, \dots, W_n$.

Recall that the solution space W_S of a single homogeneous convolution equation $S * f \equiv 0$ in $\mathscr{E}(a,b)$, where $S \in \mathscr{E}'(a,b)$, is invariant, and it admits spectral synthesis. (See [23, Theorem 2], [20, Theorem 16.4.1], and [21, Sec. 20].) By the duality principle, this means that the annihilator submodule for W_S is exactly $I_{\hat{S}} = \{f \in \hat{\mathscr{E}}'(a,b) : Z_f \geqslant Z_{\hat{S}}\}$. Thus, if a submodule I is generated by a pair of submodules of the form I_ℓ and I_g, $\ell, g \in I$, then, for a closed invariant subspace W with annihilator submodule I, this implies the representability of W in the form of the intersection of two invariant subspaces with annihilator submodules I_ℓ and I_g, i.e., W is the solution space for the system of two homogeneous convolutions equations $\check{\ell} * f = \check{g} * f = 0$. Hence, Theorem 3 can be restated in the dual form by analogy with what is done in [17, Sec. 6], and [11, Sec. 1], as follows.

Theorem 4. *If an invariant subspace in* $\mathscr{E}(a,b)$ *admits spectral synthesis, then it coincides with the solution space of a system of at most two homogeneous convolution equations.*

Remark. Proposition 5.1 can be extended without any difficulties to inductive (projective) limits of normed spaces of holomorphic functions determined by integrated norms.

Analogs of Theorem 3 can be established for the space of exponential-type entire functions which are Fourier–Laplace transforms of Beurling or Roumier type ultradisributions (see [25]), for the space of hyperfunctions (see [20, Chap. 9]) with supports on subintervals compactly embedded in a fixed interval, for the space of analytic functionals in spaces of holomorphic functions on a convex domain $D \subset \mathbb{C}$ with constraints on their growth near the boundary ∂D (see [26]), etc. It seems that such analogs can be used to obtain results similar to Theorem 4 in terms of convolution type equations if the convolution is possible or in terms of infinite-order differential equations (see [21, Q8 Secs. 21 and 22]) in spaces of Beurling or Roumier type (see [25]) ultradifferentiable functions, in spaces of real-analytic function on an interval, and in spaces of holomorphic functions in a convex domain $D \subset \mathbb{C}$ with a prescribed rate of growth near the boundary ∂D [26].

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