

**SUBSEQUENCES OF ZEROS
FOR BERNSTEIN SPACES
AND THE COMPLETENESS OF SYSTEMS OF EXPONENTIALS
IN SPACES OF FUNCTIONS ON AN INTERVAL**

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ABSTRACT. Let $\sigma > 0$. The symbol B_σ^∞ denotes the space of all entire functions of exponential type not exceeding σ that are bounded on the real axis. Various exact descriptions of uniqueness sequences for the Bernstein spaces B_σ^∞ are given in terms of σ and the Poisson and Hilbert transformations. These descriptions lead to completeness criteria for systems of exponentials (up to one or two members) in various classical function spaces on an interval (closed or open) of length d .

§1. INTRODUCTION

1.1. Main definitions, notions, and notation. We use the notation $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $\mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ for the open upper and lower half-planes of the complex plane \mathbb{C} , and put $\mathbb{C}_\pm := \mathbb{C}_- \cup \mathbb{C}_+ = \mathbb{C} \setminus \mathbb{R}$, where \mathbb{R} is the real axis. The symbol $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere; $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk centered at zero. Also, we use the symbols $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$, $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$, and $\mathbb{D}_* := \mathbb{D} \setminus \{0\}$ to denote the “punctured” real axis, complex plane, and unit disk \mathbb{D} , respectively.

For a set $S \subset \mathbb{C}_\infty$, we denote by \bar{S} and ∂S the closure and the boundary of S in \mathbb{C}_∞ , but for $S \subset \mathbb{R}$ the boundary of S in \mathbb{R} is denoted by $\partial_{\mathbb{R}} S$.

The symbol $I_d \subset \mathbb{R}$ (respectively, \bar{I}_d) will denote an open (respectively, closed) interval of length d .

Next, we denote by $\operatorname{Hol}(\Omega)$, $\operatorname{sbh}(\Omega)$ and $\operatorname{har}(\Omega)$ the classes of holomorphic, subharmonic, and harmonic functions (respectively) in a domain $\Omega \subset \mathbb{C}_\infty$.

Let

$$(1.1) \quad \Lambda = \{\lambda_k\}, \quad k = 1, 2, \dots,$$

be an at most countable sequence in a domain $\Omega \subset \mathbb{C}$ without limit points inside Ω . Some points among the λ_k may occur more than once. It is also permitted that Λ is the empty set. With each sequence Λ , we associate the nonnegative integer-valued counting measure n_Λ on Ω (the divisor) defined by

$$(1.2) \quad n_\Lambda(S) := \sum_{\lambda_k \in S} 1, \quad S \subset \Omega,$$

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i.e., its value is the number of the λ_k 's contained in S . For a point $z \in \mathbb{C}$, we put $n_\Lambda(z) := n_\Lambda(\{z\})$. We also introduce the *radial counting function* of the sequence $\Lambda \subset \mathbb{C}$ by the formula $n_\Lambda^{\text{rad}}(t) := n(t\bar{\mathbb{D}})$, $t \geq 0$. In general, operations on sequences, relations involving them, and many other notions are the same as in [1].

With a sequence $\Lambda \subset \mathbb{C}$, we associate the system of (multiple) exponentials

$$(1.3) \quad \text{Exp}^\Lambda := \{z \mapsto z^{p-1} e^{i\lambda z} : z \in \mathbb{C}, \lambda \in \Lambda, 1 \leq p \leq n_\Lambda(\lambda), p \in \mathbb{N}\},$$

where \mathbb{N} is the set of positive integers. The imaginary unit i arises in (1.3) mainly by tradition, which is related to completeness problems for systems of exponentials in spaces on subsets of the real axis (see [2, 3, 4, 5]), compare with [1]), and to the use of Fourier transforms. As usual, by a system Exp^Λ on a subset $S \subset \mathbb{C}$ we mean the system of restrictions of the functions in (1.3) to S .

Let $f \in \text{Hol}(\Omega)$, and let $f \not\equiv 0$ on Ω . The sequence of zeros of f in Ω enumerated with multiplicities will be denoted by Zero_f . A sequence Λ is called a *sequence of zeros* for a subclass $H \subset \text{Hol}(\Omega)$ if there exists a function $f \in H$ such that $\Lambda = \text{Zero}_f$, i.e., $n_\Lambda = n_{\text{Zero}_f}$. A function $f \in \text{Hol}(\Omega)$ *vanishes* on Λ if $\Lambda \subset \text{Zero}_f$, i.e., $n_\Lambda \leq n_{\text{Zero}_f}$. A sequence Λ is a *subsequence of zeros* for a subclass $H \subset \text{Hol}(\Omega)$ if there exists a nonzero function $f \in H$ that vanishes on Λ . If the class H is closed under subtraction, then a subsequence of zeros for H is also called a *nonuniqueness sequence*, and any $\Lambda \subset \Omega$ that is not a subsequence of zeros for H is also called a *uniqueness sequence* for H .

Let $\sigma > 0$. The principal ‘‘model’’ class H considered in the paper is the Bernstein space B_σ^∞ (see [2]) of all entire functions $f \in \text{Hol}(\mathbb{C})$ of exponential type at most σ (i.e., $\limsup_{z \rightarrow \infty} (\log|f(z)|)/|z| \leq \sigma$) that are bounded on the real axis, i.e., $\sup_{x \in \mathbb{R}} |f(x)| < +\infty$.

Alternatively (see [2]), the Bernstein space B_σ^∞ can be defined as the class of all entire functions subject to the restriction

$$(1.4) \quad \log|f(z)| \leq \sigma |\text{Im } z| + c_f, \quad z \in \mathbb{C},$$

where c_f is a constant.

1.2. Some classical and some known results. Because of the well-known relationship between the completeness of systems of exponentials (1.3) in function spaces on an interval I_d or a segment \bar{I}_d and uniqueness sequences in the Bernstein space $B_{d/2}^\infty$, some results will be stated as conditions for the (non)completeness of a systems of exponentials. It should be noted that many theorems in this spirit were proved or stated in [1, 2, 3, 4, 5, 6, 7, 8, 9]. We formulate and discuss some of them.

Cartwright theorem. *Suppose that for a sequence (1.1) the series*

$$(1.5) \quad \sum_{\text{Im } \lambda_k \neq 0} \left| \text{Im} \frac{1}{t - \lambda_k} \right|$$

diverges for some $t \in \mathbb{R}$. Then Λ is a uniqueness sequence for B_σ^∞ for every $\sigma > 0$, and the system Exp^Λ is complete in the spaces $C(\bar{I}_d)$ and $L^p(I_d)$ for any finite $d > 0$ and $p \geq 1$. Furthermore, if the series

$$(1.6) \quad \sum_{\text{Im } \lambda_k \neq 0} \frac{1}{|\lambda_k|^2}$$

converges, then the convergence of the series (1.5) for some particular t implies its convergence for all $t \in \mathbb{R}$.

Remark 1. The divergence of the series (1.6) and even of the series $\sum_{\lambda_k \neq 0} 1/|\lambda_k|^\alpha$ for some $\alpha > 1$, and also the condition that either the upper density

$$(1.7) \quad \limsup_{t \rightarrow +\infty} \frac{n_\Lambda^{\text{rad}}(t)}{t}$$

or the averaged upper density ($0 \notin \Lambda$)

$$(1.8) \quad \limsup_{r \rightarrow +\infty} \frac{N_\Lambda(r)}{r}, \quad N_\Lambda(r) := \int_0^r \frac{n_\Lambda^{\text{rad}}(t)}{t} dt = \sum_k \log^+ \frac{|\lambda_k|}{r},$$

is infinite have the same consequence as the divergence of the series (1.5) in the Cartwright theorem, see [5, Theorem 7].

Remark 2. For a wide class of spaces including B_σ^∞ , the sequence Λ remains to be a (non)uniqueness sequence after a shift of a finite number of points, and the (non)completeness of a sequence of exponentials Exp^Λ is stable under this transformation in the function spaces under study (see [5] and a general result in [1, Theorem 1.1.4]). Therefore, without loss of generality we may assume if necessary that $0 \notin \Lambda$ for the sequence (1.1).

We present a weak converse to the Cartwright theorem.

Schwartz theorem (see [4] and [5, Theorem 41]). *Suppose that a sequence (1.1) satisfies the following two conditions:*

$$(1.9) \quad 0 < \alpha \leq |\arg \lambda_k| \leq \pi - \alpha, \quad k \in \mathbb{N}, \quad \sum_{\lambda_k \neq 0} \left| \text{Im} \frac{1}{\lambda_k} \right| < +\infty,$$

or only the condition $\sum_{\lambda_k \neq 0} 1/|\lambda_k| < +\infty$. Then Λ is a subsequence of zeros for the spaces B_σ^∞ for all $\sigma > 0$, and the system Exp^Λ is not complete in the spaces $C(\bar{I}_d)$ and $L^p(I_d)$ for every $d > 0$.

New proofs of the Cartwright and Schwartz theorems will be presented below to illustrate the specific features of our approach.

The above theorems have a drawback. Specifically, they do not “feel” the removal from Λ or the addition to Λ of finitely many points. To a certain extent, this is overcome in Levinson’s theorems, see [3]. We state only one to illustrate them.

Levinson theorem. *Suppose that $p \in [1, +\infty]$, and the sequence (1.1) has the property that the characteristic*

$$(1.10) \quad \limsup_{r \rightarrow \infty} \left(N_\Lambda(r) - \frac{d}{\pi} r + \frac{\log r}{p} \right)$$

is equal to $+\infty$ for $p = 1, +\infty$ or is not equal to $-\infty$ for $1 < p < +\infty$. Then the system Exp^Λ is complete in $L^p(I_d)$ for $p \in [1, +\infty)$ and in $C(\bar{I}_d)$ for $p = +\infty$. In (1.10), the number p cannot be replaced by a smaller one.

The most substantial generalization of the Levinson theorem was published recently in a joint paper by Makarov and Poltoratskii in [10, §2.11]. In that paper, some general criteria were proved (see [10, §3.2, Proposition, Theorem, and Example]), which describe the nonuniqueness sets for a wide class of model spaces. Later, these criteria were generalized by Baranov, see [12, Theorems 9.1.1 and 9.1.2]. A certain criterion for a sequence to be one of uniqueness for a Bernstein space can also be deduced from those results. We do not state it here for two reasons. First, it requires some preparations. Second, though the Hilbert transformation is employed both in the results by Makarov, Poltoratskii, and Baranov and in our main theorem, this is done in opposite ways in a sense. In the Makarov–Poltoratskii–Baranov theorems, it is required *to construct* (or to

prove the existence of) certain *special functions* for the given sequence Λ and the space in question; the Hilbert transform of one of them must possess certain specific properties. But under our approach, *the verification of an infinite series of uniform integral estimates is required for a special class of test functions* that are defined in terms of the Hilbert transformation.

The following classical theorem stays until now a deepest result on the completeness of systems of exponentials and on (non)uniqueness sequences for the Bernstein spaces.

Beurling–Malliavin theorem ([5, Theorem 77], Redheffer’s interpretation). *Suppose that $\lambda_k \neq 0$ in the sequence (1.1). If there exists a number $c > 0$ and a sequence $\{n_k\}_{k \in \mathbb{N}}$ of pairwise different integers such that the series $\sum_{k \in \mathbb{N}} \left| \frac{1}{\lambda_k} - \frac{c}{2\pi n_k} \right|$ converges, then Λ is a subsequence of zeros for $B_{d/2}^\infty$ and the system Exp^Λ is not complete in $C(\bar{I}_d)$ and in $L^p(I_d)$, $p \geq 1$, whenever $d > c$. Conversely, if this series diverges for every sequence $\{n_k\}_{k \in \mathbb{N}}$ of pairwise different integers, then Λ is a uniqueness sequence for $B_{d/2}^\infty$ and the system Exp^Λ is complete in $C(\bar{I}_d)$ and in $L^p(I_d)$ for every $d < c$.*

The property to be a (non)uniqueness sequence as well as the property to be a (non)complete sequence of exponentials may happen to be stable under small variations of the sequence (respectively, under small shifts of the exponents Λ in (1.3)). Each stability statement of this sort yields immediately conditions sufficient for (non)uniqueness or (non)completeness if we start with a sequence Λ for which the required property is already known. The following statement is the origin of many such results.

Redheffer–Aleksander theorem. *If two sequences*

$$(1.11) \quad \Lambda := \{\lambda_k\}, \quad \Gamma := \{\gamma_k\}, \quad k = 1, 2, \dots,$$

satisfy the condition $\sum_{k=1}^\infty \frac{|\lambda_k - \gamma_k|}{1 + |\text{Im } \lambda_k| + |\text{Im } \gamma_k|} < \infty$, then, for any $d > 0$, the systems Exp^Λ and Exp^Γ are simultaneously complete or not in each of the spaces $C(\bar{I}_d)$ and $L^p(I_d)$, $1 \leq p < \infty$.

The following theorem was obtained by Baranov not very long ago.

Baranov theorem (see [11, Theorem 2.3], [12, Theorem 9.2.1]). *If a sequence (1.11) in the upper half-plane \mathbb{C}_+ satisfies the condition $\sup_{t \in \mathbb{R}} \sum_k \frac{|\lambda_k - \gamma_k|}{|t - \lambda_k - i|} < \infty$, then $\{\lambda_k\}$ and $\{\gamma_k\}$ may be nonuniqueness sequences for B_σ^∞ only simultaneously.*

Baranov’s theorem is also true for the Paley–Wiener spaces PW_σ^p .

§2. PRINCIPAL RESULTS

2.1. Criterion for being a subsequence of zeros for Bernstein spaces. Recall that, for a function $\varphi \in L^1(\mathbb{R})$, $\varphi: \mathbb{R}_* \rightarrow \mathbb{R}$, its *direct Hilbert transform* H is defined by the integral (see [13, 14, 15])

$$(\mathsf{H}\varphi)(x) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{\varphi(t)}{x - t} dt, \quad x \in \mathbb{R}_*,$$

where the dashed integral denotes the Cauchy principal value. This function $\mathsf{H}\varphi$ is defined a.e. on \mathbb{R} . The *inverse Hilbert transform* differs¹ only by the sign:

$$(2.1) \quad (\mathsf{H}^{-1}\varphi)(x) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{\varphi(t)}{t - x} dt = -(\mathsf{H}\varphi)(x), \quad x \in \mathbb{R}_*.$$

¹Sometimes, especially in applications to engineering, the terms are used in the opposite sense.

Classes RP_0^m of test functions. In what follows, we *only consider the case where*

$$(2.2) \quad m \in (\mathbb{N} \setminus \{1\}) \cup \{\infty\}.$$

Definition 1. We introduce the class RP_0^m of test functions as the subclass of all *positive continuous functions*

$$(2.3c) \quad \varphi: \mathbb{R}_* \rightarrow [0, +\infty), \quad \varphi \in C(\mathbb{R}_*),$$

$$(2.3z) \quad \text{with the zero set } Z_\varphi := \{x \in \mathbb{R}_* : \varphi(x) = 0\}$$

$$(2.3p) \quad \text{such that for some set } E_\varphi \subset \mathbb{R}_* \text{ polar in } \mathbb{C} \text{ and closed in } \mathbb{R}_*,$$

the restriction of φ to $\mathbb{R} \setminus (\{0\} \cup Z_\varphi \cup E_\varphi) = \mathbb{R}_* \setminus (Z_\varphi \cup E_\varphi)$ belongs to the class

$$C^m(\mathbb{R} \setminus (\{0\} \cup Z_\varphi \cup E_\varphi)) = C^m(\mathbb{R}_* \setminus (Z_\varphi \cup E_\varphi)),$$

and the following conditions are fulfilled:

- *the compact support requirement*

$$(2.4) \quad \varphi(x) \equiv 0, \quad |x| \geq R_\varphi > 0,$$

where $R_\varphi > 0$ is a constant depending on φ ;

- *seminormalization at zero*

$$(2.5) \quad \limsup_{0 \neq x \rightarrow 0} \frac{\varphi(x)}{-\log|x|} \leq 1;$$

- *conjugate positivity condition*

$$(2.6) \quad (-H\varphi)'(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\varphi(t) - \varphi(x)}{(t-x)^2} dt \geq 0, \quad x \in \mathbb{R}_* \setminus (Z_\varphi \cup E_\varphi);$$

after a change of variables, this inequality can be rewritten in the form

$$(2.7) \quad (-H\varphi)'(x) = \frac{1}{\pi} \int_0^{+\infty} \frac{\varphi(x+t) + \varphi(x-t) - 2\varphi(x)}{t^2} dt \geq 0;$$

next, equality in (2.6)–(2.7) is ensured by the last identity in (4.7) (see the final part of Proposition 2 proved below).

By (2.1) and (2.6), the conjugate positivity conditions (2.6)–(2.7) are equivalent to the requirement that *the inverse Hilbert transform $H^{-1}\varphi = -H\varphi$ be monotone increasing²* on each connected component of the complement to \mathbb{R}_* of the set $Z_\varphi \cup E_\varphi$. Since this set is closed in \mathbb{R}_* , all connected components mentioned above are intervals.

The class RP_0^m of test functions is invariant under homothety, which will be useful in the sequel (see §6): *if φ belongs to RP_0^m , then for every $r \in \mathbb{R}_*$ the function*

$$(2.8) \quad \varphi(\cdot/r): t \mapsto \varphi(t/r), \quad t \in \mathbb{R}_*, \text{ also belongs to } RP_0^m.$$

The left-hand sides of (2.6)–(2.7) can be replaced by the direct Hilbert transform of the derivative φ' because $(-H\varphi)' = -H\varphi' = H^{-1}\varphi'$, provided φ' belongs to $L^p(\mathbb{R})$ with some $p > 1$ (see [13, Subsection 4.8]), or provided we treat the derivatives and Hilbert transforms in the sense of the theory of distributions (see [15, §3.3]).

²A function f is monotone increasing on $I \subset \mathbb{R}$ if $f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$, $x_1, x_2 \in I$.

Poisson transformation. For $\lambda \in \mathbb{C}_\pm := \mathbb{C} \setminus \mathbb{R}$, the *Poisson integral* $P_{\mathbb{C}_\pm} \varphi$ of a function $\varphi \in L^1(\mathbb{R})$ is defined by

$$(2.9) \quad (P_{\mathbb{C}_\pm} \varphi)(\lambda) := \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{|\operatorname{Im} \lambda|}{(t - \operatorname{Re} \lambda)^2 + (\operatorname{Im} \lambda)^2} \varphi(t) dt, \quad \operatorname{Im} \lambda \neq 0.$$

For $\lambda \in \mathbb{R}_*$, we put

$$(2.10) \quad (P_{\mathbb{C}_\pm} \varphi)(\lambda) := \varphi(\lambda), \quad \lambda \in \mathbb{R}_*.$$

The Poisson integral in (2.9) is a harmonic function on $\mathbb{C} \setminus \mathbb{R}$. Taken together, the definitions (2.9) and (2.10) determine the *Poisson transformation* $P_{\mathbb{C}_\pm}$.

Theorem 1 (on uniqueness sequences for the Bernstein spaces). *Let $\Lambda = \{\lambda_k\}$ be a sequence in \mathbb{C} with $0 \notin \Lambda$, and let $\sigma \in (0, +\infty)$. The following three statements are equivalent:*

- 1) $\Lambda = \{\lambda_k\}_{k \in \mathbb{N}}$ is a uniqueness sequence for the Bernstein space B_σ^∞ ;
- 2) for some (for arbitrary) m in (2.2) we have

$$(2.11) \quad \sup_{\varphi \in R\mathcal{P}_0^m} \left(\sum_{k \in \mathbb{N}} (P_{\mathbb{C}_\pm} \varphi)(\lambda_k) - \frac{\sigma}{\pi} \int_{-\infty}^{+\infty} \varphi(t) dt \right) = +\infty;$$

- 3) condition (2.11) is fulfilled with the replacement of the supremum over $R\mathcal{P}_0^m$ by the supremum over the smaller class of smooth test functions $R\mathcal{P}_0^\infty \cap C^\infty(\mathbb{R}_*)$.

Remark 2 in Subsection 1.2 shows that the condition $0 \notin \Lambda$ does not lead to any loss of generality.

Remark 3. If $\Lambda \subset \mathbb{R}_*$ is a real sequence, then (2.11) becomes very simple:

$$(2.12) \quad \sup_{\varphi \in R\mathcal{P}_0^m} \left(\sum_{\lambda \in \Lambda} \varphi(\lambda) - \frac{\sigma}{\pi} \int_{-\infty}^{+\infty} \varphi(t) dt \right) = +\infty.$$

Remark 4. As was mentioned in Remark 1 (see Subsection 1.2), Theorem 1 is interesting only in the case where the series (1.5) converges for some $t \in \mathbb{R}$, and the sequence Λ has finite upper density. Note that every sequence Λ of finite upper density has convergence exponent of at most 1, which means the convergence of every series $\sum_{\lambda_k \neq 0} 1/|\lambda_k|^\alpha$ for every $\alpha > 1$. In particular, the series (1.6) converges. Then the series $\sum_{\operatorname{Im} \lambda_k \neq 0} \left| \operatorname{Im} \frac{1}{\lambda_k} \right|$ also converges. Moreover, the series (1.5) converges uniformly in $t \in [-r, r] \subset \mathbb{R}$ for every fixed $r > 0$. Indeed, let $t \in [-r, r]$. Then, for every $R' > R \geq 2r$, it is easy to deduce the inequality

$$(2.13) \quad \sum_{\substack{\operatorname{Im} \lambda_k \neq 0 \\ R \leq |\lambda_k| \leq R'}} \left| \operatorname{Im} \frac{1}{t - \lambda_k} \right| \leq \sum_{\substack{\operatorname{Im} \lambda_k \neq 0 \\ bR \leq |\lambda_k| \leq R'}} \left| \operatorname{Im} \frac{1}{\lambda_k} \right| + \sum_{R \leq |\lambda_k| \leq R'} \frac{2r}{|\lambda_k|^2},$$

where the right-hand side tends to zero as $R', R \rightarrow +\infty$ independently of $t \in [-r, r]$. Now, the uniform convergence follows from the Cauchy criterion. Since every test function has compact support (included in $[-R_\varphi, R_\varphi]$), in the sum $\sum_{k \in \mathbb{N}} (P_{\mathbb{C}_\pm} \varphi)(\lambda_k)$ we can interchange summation and the (Poisson) integral.³ Consequently, if the series (1.5) converge (for some t) and we have (1.6), then

$$\sum_{k \in \mathbb{N}} (P_{\mathbb{C}_\pm} \varphi)(\lambda_k) = \int_{\mathbb{R}} \left(\sum_{\operatorname{Im} \lambda_k \neq 0} \frac{1}{\pi} \left| \operatorname{Im} \frac{1}{t - \lambda_k} \right| \right) \varphi(t) dt + \sum_{\operatorname{Im} \lambda_k = 0} \varphi(\lambda_k)$$

so that the sum in (2.11) can be replaced by the right-hand side of the last identity.

³The possibility of this interchange also follows from the positivity of the terms of the series, see [16, Chapter IV, §4, Corollary 4].

Remark 5. In conditions (2.11) and (2.12), a direct analogy can be observed with the definitions of the natural order relations for real Radon measures or for distributions. Indeed, if ν and μ are two such measures or distributions on \mathbb{C} , then, by definition, $\nu \leq \mu$ means that

$$(2.14) \quad \sup_{\varphi \in (C_0^\infty(\mathbb{C}))^+} (\nu(\varphi) - \mu(\varphi)) \leq 0,$$

where $(C_0^\infty(\mathbb{C}))^+$ is the class of all compactly supported infinitely differentiable positive functions on \mathbb{C} .

On the other hand, every sequence $\Lambda = \{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{C}$ of points gives rise to the counting measure n_Λ defined by the rule (1.2). The measure μ_σ on \mathbb{R} with the density $d\mu_\sigma(x) := \frac{\sigma}{\pi} dx$, $x \in \mathbb{R}$, is the Riesz measure of the subharmonic function $M_\sigma(z) := \sigma |\operatorname{Im} z|$, $z \in \mathbb{C}$, $\mu_\sigma := \frac{1}{2\pi} \Delta M_\sigma \geq 0$, where Δ stands for the Laplace operator on Schwartz distributions. The function M_σ occurs on the right in (1.4) and determines the Bernstein class B_σ^∞ . In this notation, (2.11) and (2.12) can be rewritten in the form

$$\sup_{\varphi \in RP_0^m} (n_\Lambda(P_{\mathbb{C}_\pm} \varphi) - \mu_\sigma(\varphi)) < +\infty.$$

It is useful to compare this formula with relation (2.14). An analogy with that relation is fairly transparent, especially in item 3), where we use infinitely differentiable test functions belonging to $RP_0^\infty \cap C^\infty(\mathbb{R}_*)$.

2.2. Completeness with excess 0 or 1 for systems of exponentials in function spaces on an interval.

In this subsection, we employ the terminology and notation from [1] (sometimes with slight modifications). We recall some definitions. A system of vectors in a locally convex space E is *complete* if the closure of its linear hull coincides with E . A system of vectors is *minimal* in E if no vector in the system belongs to the closure of the linear hull of the other vectors. A system is said to be *exact* if it is complete and minimal. Let \mathcal{E}_Λ be a sequence of pairwise different vectors in $\mathcal{E} \subset E$, and suppose that it is indexed by points of $\Lambda = \{\lambda_k\} \subset \mathbb{C}$. The system \mathcal{E}_Λ is said to have an *excess* $\operatorname{exc} \mathcal{E}_\Lambda = q \in \mathbb{Z}$ in E (relative to \mathcal{E}) if a system exact in E arises

- after removal of q vectors from \mathcal{E}_Λ if $q \geq 0$;
- after addition of $|q|$ new vectors taken from \mathcal{E} to \mathcal{E}_Λ if $q < 0$.

If completeness is not destroyed (does not arise) after removal (respectively, addition) of an arbitrary finite collection of pairwise different vectors from \mathcal{E}_Λ (respectively, vectors belonging to $\mathcal{E} \setminus \mathcal{E}_\Lambda$), we put $\operatorname{exc} \mathcal{E}_\Lambda := +\infty$ (respectively, $\operatorname{exc} \mathcal{E}_\Lambda := -\infty$).

In the sequel, when talking about the excess of a system of exponentials $\operatorname{Exp}^\Lambda$ in a function space, we shall write $\operatorname{exc} \Lambda$ in place of $\operatorname{exc} \operatorname{Exp}^\Lambda$. For the spaces under study, the excess for multiple exponentials is well defined because it does not depend on the exponentials that are removed or added (general results on this matter can be found in [1, Theorem 1.1.4(3), Subsection 1.1.3]).

Theorem 2 (on the completeness of a system of exponentials on intervals). *If a sequence $\Lambda \subset \mathbb{C}$ of points, $0 \notin \Lambda$, satisfies (2.11) or, under the assumption $\Lambda \subset \mathbb{R}$, it satisfies (2.12), then the system $\operatorname{Exp}^\Lambda$ is complete in any space $C(\bar{I}_{2\sigma})$ or $L^p(I_{2\sigma})$, $p \geq 1$. Conversely, if the left-hand side in (2.11) or, under the restriction $\Lambda \subset \mathbb{R}$, in (2.12), is finite, then, relative to the system of all multiple exponentials, we have $\operatorname{exc} \Lambda \leq 0$ for $C(\bar{I}_{2\sigma})$ and for $L^p(I_{2\sigma})$, where $p \geq 2$; moreover, $\operatorname{exc} \Lambda \leq 1$ for $L^p(I_{2\sigma})$ provided $1 \leq p < 2$.*

Furthermore, in (2.11) or (2.12), the classes RP_0^m of test functions can be replaced (in any situation) by the smaller class that occurs in item 3) of Theorem 1.

Remark 6. In a somewhat different and simpler form, Theorems 1 and 2 were announced in the 3rd and 4th edition of the book [1], see Theorem 2.1.12 and Corollary 2.1.2 therein, and also in the short survey [17, Theorem 4 and Corollary 2]. It should also be noted that when we want to deduce from Theorem 1 various conditions sufficient for uniqueness (see Subsection 6.1 below) and also conditions of completeness for a system of exponentials, it is useful to possibly extend the set of test functions (see the implication $2) \implies 1)$ in Theorem 1 and the first part of Theorem 2). Conversely, when we study conditions sufficient for nonuniqueness or for noncompleteness of systems of exponentials, it is convenient to possibly restrict the class of test functions without destruction of Theorem 1 (see the implication $3) \implies 1)$) or the second part of Theorem 2 on completeness (see the final statement in it). Thus, both extension and restriction of the class of test functions are of interest. At the same time, the problem of restriction of the class of test functions in these theorems is much subtler and much more complicated than the problem of extending the class RP_0^2 . We are going to return to this elsewhere.

§3. DESCRIPTION OF UNIQUENESS SEQUENCES FOR BERNSTEIN SPACES IN TERMS OF THE JENSEN POTENTIALS

We shall need some definitions and facts from the paper [18] and the survey [1]. In the sequel, the symbol \Subset denotes a precompact inclusion.

Definition 2. A function V subharmonic in \mathbb{C}_* is called a *Jensen potential*⁴ (with a pole at 0) if the following three conditions are fulfilled:

- 1) $V(\zeta) \geq 0$ for $\zeta \in \mathbb{C}_*$ (positivity);
- 2) there exists a compact set $K \Subset \mathbb{C}$ such that $V \equiv 0$ on $\mathbb{C} \setminus K$ (compact support);
- 3) $\limsup_{\zeta \rightarrow 0} \frac{V(\zeta)}{-\log|\zeta|} \leq 1$ (seminormalization at zero).

The class of all Jensen potentials will be denoted by \mathcal{P}_{J_0} .

In fact, the seminormalization condition for V at zero gives more: *there exists a number $R_V > 0$ such that*

$$V(\zeta) \leq \log^+ \frac{R_V}{|\zeta|}, \quad \zeta \in \mathbb{C}_*$$

(see [18, Proposition 1.5]), and if $V(\infty) := 0$, then $V \in \text{sbh}(\mathbb{C}_\infty \setminus \{0\})$.

The role played by the Jensen potentials in the proof of Theorem 1 is dictated by a result proved in [18, Theorem 2]. Let $M \in \text{sbh}(\mathbb{C})$ with $M \not\equiv -\infty$, and let $\nu_M := \frac{1}{2\pi} \Delta M$ be the Riesz measure for M , where Δ is the Laplace operator on distributions.

We introduce the weighted space of entire functions

$$\text{Hol}(\mathbb{C}; M) := \{f \in \text{Hol}(\mathbb{C}) : \sup_{z \in \mathbb{C}} \frac{|f(z)|}{\exp M} < +\infty\}.$$

Theorem A (see [18, Introduction, Theorem 2]). *Let $M \in \text{sbh}(\mathbb{C})$. If there exists a constant $C \in \mathbb{R}$ such that*

$$(3.1) \quad \sup_{|z-\zeta| \leq 1} M(\zeta) \leq M(z) + C \quad \text{for all } z \in \mathbb{C},$$

then the following statements are equivalent:

- 1) *the sequence $\Lambda = \{\lambda_k\} \subset \mathbb{C}$ is a uniqueness sequence for $\text{Hol}(\mathbb{C}; M)$;*

⁴In [18, 1] and other our papers the term ‘‘Jensen function’’ was used, but it turned out that it had been employed before in the theory of generalized analytic functions (see the references in [19]). This forced us to pass to the term ‘‘Jensen potential’’.

2) we have

$$(3.2) \quad \sup_{V \in \mathcal{P}_{J_0}} \left(\sum_k V(\lambda_k) - \int V d\nu_M \right) = +\infty.$$

It has been known since long ago (see [2]) that the Bernstein space B_σ^∞ coincides with $\text{Hol}(\mathbb{C}; M)$ for $M(z) \equiv \sigma |\text{Im } z|$, $z \in \mathbb{C}$, and, clearly, such a subharmonic function M , whose Riesz measure's density is

$$(3.3) \quad d\nu_M(t) = \frac{\sigma}{\pi} dt, \quad t \in \mathbb{R}$$

(where dt is the Lebesgue measure on \mathbb{R}), satisfies (3.1) with $C = \sigma$.

Recall that the Jensen measures μ are defined as compactly supported nonnegative measures that satisfy the condition

$$u(0) \leq \int u d\mu, \quad u \in \text{sbh}(\mathbb{C}).$$

We denote this class of measures by J_0 . The class of all Jensen potentials \mathcal{P}_{J_0} coincides (see [18, §4, Proposition 4.1]) with the class of functions

$$(3.4) \quad V_\mu(\zeta) := \int \log \left| 1 - \frac{z}{\zeta} \right| d\mu(z) = \int \log |\zeta - z| d\mu(z) - \log |\zeta|, \quad \mu \in J_0.$$

By the proof of the main theorem in [18, §2, Main theorem]⁵ and by [18, §2, Remark 1], in that main theorem we may restrict ourselves to smooth (with densities of class $C^\infty(\mathbb{C})$) Jensen measures supported outside an arbitrarily fixed disk $R\mathbb{D}$. By the definition (3.4), such measures give rise to Jensen potentials smooth in \mathbb{C}_* and harmonic in $R\mathbb{D}_*$, see [18, §4, Example 2]. Theorem A is deduced from [18, §2, Main theorem] with the help of [18, §5, Theorem 2], where Jensen measures are replaced by Jensen potentials. Thus, also in Theorem A smooth Jensen potentials harmonic in $R\mathbb{D}_*$ would suffice. This yields the following statement.

Remark 7. In formula (3.2) (see Theorem A) the term $\sup_{V \in \mathcal{P}_{J_0}}$ can be replaced by the supremum over all infinitely differentiable Jensen potentials $V \in \mathcal{P}_{J_0} \cap C^\infty(\mathbb{C}_*)$ with harmonic restriction to $R\mathbb{D}_*$.

In the special case where $\nu_1 = n_\Lambda$, $d\nu_2 = \nu_M$, and $M(z) \equiv \sigma |\text{Im } z|$, formula (3.3) and Remark 7 allow us to restate Theorem A as follows.

Theorem 3. *Let $\Lambda = \{\lambda_k\}$ be a sequence in \mathbb{C} , and let $\sigma \in (0, +\infty)$. The following three statements are equivalent:*

- 1) $\Lambda = \{\lambda_k\} \subset \mathbb{C}$ is a uniqueness sequence for the Bernstein space B_σ^∞ ;
- 2) for some (for any) $R > 0$ we have

$$(3.5) \quad \sup \left\{ \sum_k V(\lambda_k) - \frac{\sigma}{\pi} \int_{-\infty}^{+\infty} V(t) dt : V \in \mathcal{P}_{J_0} \cap C^\infty(\mathbb{C}_*), V \Big|_{R\mathbb{D}_*} \in \text{har}(R\mathbb{D}_*) \right\} = +\infty;$$

- 3) the following condition holds true:

$$(3.6) \quad \sup_{V \in \mathcal{P}_{J_0}} \left(\sum_k V(\lambda_k) - \frac{\sigma}{\pi} \int_{-\infty}^{+\infty} V(t) dt \right) = +\infty.$$

⁵In the proof of that theorem, there are some typos in item 2a). To fix them, it suffices to replace the product $\varphi_\Lambda g_R$ by $\varphi_\Lambda \exp g_R$ throughout. However, the same correction was already made in the more recent and informative statement [20, Proof of Proposition 9.2, item (a)].

§4. EXTENSION OF TEST FUNCTIONS

The following statement will play a principal role in this section.

Proposition 1. *Suppose that \mathcal{O} is an open subset of \mathbb{C} and a function $V: \mathcal{O} \rightarrow \mathbb{R}$ is continuous on \mathcal{O} , $V \in \text{har}(\mathcal{O} \setminus \mathbb{R})$, and*

$$V \in C^1(\bar{\mathbb{C}}_+ \cap \mathcal{O}), \quad V \in C^1(\bar{\mathbb{C}}_- \cap \mathcal{O}).$$

The function V is subharmonic on \mathcal{O} if and only if

$$(4.1) \quad \left(\frac{\partial V}{\partial \bar{n}_+^{\text{in}}} + \frac{\partial V}{\partial \bar{n}_-^{\text{in}}} \right) (x) := \lim_{0 < y \rightarrow 0} \frac{V(x+iy) - V(x)}{y} \\ + \lim_{0 > y \rightarrow 0} \frac{V(x+iy) - V(x)}{-y} \geq 0, \quad x \in \mathbb{R} \cap \mathcal{O},$$

where $\frac{\partial}{\partial \bar{n}_+^{\text{in}}}$ and $\frac{\partial}{\partial \bar{n}_-^{\text{in}}}$ denote differentiations in the direction of the inner unit normal of $\mathbb{C}_+ \cap \mathcal{O}$ and $\mathbb{C}_- \cap \mathcal{O}$, respectively. Moreover, the density ν_V of the Riesz measure of the function $V \in \text{sbh}(\mathcal{O})$ with support $\text{supp } \nu_V \subset \mathbb{R} \cap \mathcal{O}$ coincides with the left-hand side of inequality (4.1) divided by 2π .

Moreover, if under condition (4.1) the function V is also defined as positive and continuous in a neighborhood \mathcal{O}_0 (open in \mathbb{C}) of the union $A_0 := Z_0 \cup E_0$ of two closed subsets Z_0 and E_0 of $\mathbb{R} \setminus \mathcal{O}$, is subharmonic on $\mathcal{O}_0 \setminus A_0$, and satisfies $V(x) \equiv 0$ for $x \in Z_0$, and E_0 is a polar set in \mathbb{C} , then V is subharmonic in the open neighborhood $\mathcal{O} \cup \mathcal{O}_0$ of $(\mathbb{R} \cap \mathcal{O}) \cup A_0 \subset \mathbb{R}$.

Proof. Only the case where $\mathbb{R} \cap \mathcal{O} \neq \emptyset$ is nontrivial.

Lemma A. *Let D be a domain in \mathbb{C} , and let S be a C^1 -curve that divides D in two subdomains D_1 and D_2 .*

Suppose that $V \in C_{\mathbb{R}}(D) \cap C^2(D_1 \cup D_2)$ and $V \in \text{sbh}(D_k)$, $k = 1, 2$. If

$$V \Big|_{D_k \cup S} =: V_k \in C^1(D_k \cup S), \quad k = 1, 2,$$

then the function is subharmonic in D if and only if

$$(4.2) \quad \frac{\partial V_1}{\partial \bar{n}_1^{\text{in}}} + \frac{\partial V_2}{\partial \bar{n}_2^{\text{in}}} \geq 0$$

on S , where the $\frac{\partial}{\partial \bar{n}_k^{\text{in}}}$ stand for differentiations in the direction of the inner unit normal in $D_k \cup S$, $k = 1, 2$.

To prove the criterion for V to be subharmonic on \mathcal{O} , it suffices to directly apply Lemma A. (After [21, Lemma 3.8.1], this lemma was reproved independently by Blanchet in [22, Theorem 3.1], see also [23, Lemma 4.1]). For that, we must put $D := \mathcal{O}$, $D_1 := \mathcal{O} \cap \mathbb{C}_+$, $D_2 := \mathcal{O} \cap \mathbb{C}_-$, $S := \mathbb{R} \cap \mathcal{O}$ and rewrite (4.2) in the more explicit form (4.1).

We describe the Riesz measure of the function $V \in \text{sbh}(\mathcal{O})$. For the role of the domain D , now we take an arbitrary disk $x_0 + r\mathbb{D} \Subset \mathcal{O}$. We consider the open interval $(x_0 - r, x_0 + r) \Subset \mathcal{O} \cap \mathbb{R}$ and the upper and lower open half-disks

$$(4.3) \quad D_+ := (x_0 + r\mathbb{D}) \cap \mathbb{C}_+, \quad D_- := (x_0 + r\mathbb{D}) \cap \mathbb{C}_-$$

in \mathcal{O} .

Applying the Green formula twice in each half-disk D_{\pm} to a positive infinitely differentiable function $\psi \in (C_0^\infty(D))^+$ compactly supported in D and to the function V harmonic D_{\pm} , and adding the results, we obtain

$$\int_D V \Delta \psi = \int_{x_0-r}^{x_0+r} \psi(x) \left(\frac{\partial V}{\partial \bar{n}_+^{\text{in}}} + \frac{\partial V}{\partial \bar{n}_-^{\text{in}}} \right) (x) dx,$$

where the right-hand side is positive by (4.1). It follows that after division by 2π , (4.1) gives the Riesz measure ν_V for the function V subharmonic in \mathcal{O} .

If V is positive and continuous on \mathcal{O}_0 and is subharmonic on the open set $\mathcal{O}_0 \setminus A_0$, at every point $x_0 \in Z_0$ we have the mean-value inequality

$$0 = V(x_0) \leq \frac{1}{2\pi i} \int_0^{2\pi} V(x_0 + te^{i\theta}) d\theta, \quad x_0 + t\mathbb{D} \Subset \mathcal{O}_0.$$

Consequently, V is subharmonic in an open neighborhood $\mathcal{O} \cup (\mathcal{O}_0 \setminus E_0)$ of the polar set E_0 and is locally upper bounded on $\mathcal{O} \cup \mathcal{O}_0 \supset E_0$. By the theorem on removable singularities of subharmonic functions (see [24, Theorem 3.6.1]), the restriction of V to $\mathcal{O} \cup \mathcal{O}_0 \supset E_0$ admits a unique subharmonic extension to the entire set $\mathcal{O} \cup \mathcal{O}_0$; this extension is given by

$$\limsup_{\substack{z \rightarrow x, \\ z \in \mathcal{O}_0 \setminus E_0}} V(z), \quad x \in E_0.$$

Since V is continuous on \mathcal{O}_0 , the last upper limit coincides with $V(x)$, $x \in E_0$, which completes the proof of Proposition 1. \square

4.1. Extension of test functions to \mathbb{C}_* . Let $m \geq 2$. Given a test function $\varphi \in RP_0^m$, we construct a function on \mathbb{C}_* (see (2.9)–(2.10)):

$$(4.4) \quad V^\varphi(\zeta) := (\mathbb{P}_{\mathbb{C}_\pm} \varphi)(\zeta), \quad \zeta \in \mathbb{C}_*.$$

Proposition 2. *The function V^φ is positive and subharmonic in the punctured plane \mathbb{C}_* , and its restriction $V^\varphi \Big|_{\mathbb{R}_*}$ coincides with φ ; next, V^φ satisfies the condition of semi-normalization at zero:*

$$(4.5) \quad \limsup_{\substack{\zeta \rightarrow 0, \\ \zeta \in \mathbb{C}_*}} \frac{V^\varphi(\zeta)}{-\log |\zeta|} \leq 1.$$

Furthermore, the constant R_φ in (2.4) obeys the estimate

$$(4.6) \quad V^\varphi(\zeta) \leq \text{const.} \left| \text{Im} \frac{1}{\zeta} \right|, \quad |\zeta| \geq 2R_\varphi.$$

Moreover, we have

$$(4.7) \quad \begin{aligned} & \left(\frac{\partial V^\varphi}{\partial \bar{n}_+^{\text{in}}} + \frac{\partial V^\varphi}{\partial \bar{n}_-^{\text{in}}} \right) (x) \\ & := \lim_{0 < y \rightarrow 0} \frac{V^\varphi(x + iy) - V^\varphi(x)}{y} + \lim_{0 > y \rightarrow 0} \frac{V^\varphi(x + iy) - V^\varphi(x)}{-y} \\ & = 2 \lim_{0 < y \rightarrow 0} \frac{V^\varphi(x + iy) - \varphi(x)}{y} \\ & = \frac{1}{\pi} \int_{\mathbb{R}_*} \frac{\varphi(t) - \varphi(x)}{(t - x)^2} dt = (-\mathbb{H}\varphi)'(x) \geq 0, \\ & \quad x \in \mathbb{R}_* \setminus (E_\varphi \cup \partial_{\mathbb{R}} Z_\varphi), \end{aligned}$$

where Z_φ and E_φ are the sets occurring in (2.3z)–(2.3p), the left-hand side is the density for the restriction to $\mathbb{R}_* \setminus (E_\varphi \cup \partial_{\mathbb{R}} Z_\varphi)$ of the Riesz measure for the function $V^\varphi \in \text{sbh}(\mathbb{C}_*)$.

Proof. Since φ is positive, V^φ is positive everywhere except zero. Next, V^φ is continuous at the points of continuity of φ and

$$(4.8) \quad \lim_{\substack{\zeta \rightarrow x_0, \\ \zeta \in \mathbb{C}_*}} V^\varphi(\zeta) = \varphi(x_0), \quad x_0 \in \mathbb{R}_*,$$

by the well-known properties of the Poisson integral (see [25, Chapter I, Lemma 3.3]). Moreover, V^φ is infinitely differentiable in \mathbb{C}_\pm because it is harmonic in \mathbb{C}_\pm . We need the following lemma, which is stated here only for the case of the plane \mathbb{C} .

Lemma B (see [26, Chapter I, §8, Theorem]). *Suppose that a two times continuously differentiable curve Σ is defined in a neighborhood of the point $z_0 \in \mathbb{C}$. Suppose that, on one side of Σ , a harmonic function V is defined in some neighborhood of z_0 (i.e., in a connected component of the set $(z_0 + r\mathbb{D}) \cap (\mathbb{C} \setminus \Sigma)$ with r sufficiently small). Suppose also that for every $z \in \partial((z_0 + r\mathbb{D}) \cap \Sigma)$ the limit $\lim_{\zeta \rightarrow z} V(\zeta) = \psi(z)$ exists, and $\psi \in C^2(\Sigma)$. Then the gradient $\text{grad } V$ has a finite limit at every point of the boundary $\partial((z_0 + r\mathbb{D}) \cap \Sigma)$ and can be extended by continuity to $(z_0 + r\mathbb{D}) \cap \Sigma$.*

Let $x_0 \in \mathbb{R}_* \setminus (E_\varphi \cup \partial_{\mathbb{R}} Z_\varphi)$, in the notation of formula (2.3) in Definition 1. We choose a number $r > 0$ so small that

$$\Sigma := (x_0 - r, x_0 + r) \subset \mathbb{R}_* \setminus (E_\varphi \cup \partial_{\mathbb{R}} Z_\varphi).$$

By Lemma B, in the notation of (4.3), the corresponding restrictions of V belong to the classes

$$(4.9) \quad C^1(\bar{D}_+ \setminus \{x_0 - r, x_0 + r\}), \quad C^1(\bar{D}_- \setminus \{x_0 - r, x_0 + r\}),$$

where \bar{D}_+ and \bar{D}_- are the closures of D_+ and D_- in \mathbb{C} .

Since the point $x_0 \in \mathbb{R}_* \setminus (E_\varphi \cup \partial_{\mathbb{R}} Z_\varphi)$ can be chosen arbitrarily, the function $V := V^\varphi$ satisfies the conditions of Proposition 1 with $\mathcal{O} := \mathbb{C}_* \setminus (E_\varphi \cup \partial_{\mathbb{R}} Z_\varphi)$. To prove that $V = V^\varphi$ is subharmonic in $\mathbb{C}_* \setminus (E_\varphi \cup \partial_{\mathbb{R}} Z_\varphi)$, it suffices to verify condition (4.1).

The function $V^\varphi(x + iy)$, $\zeta = x + iy$, $x, y \in \mathbb{R}$, is odd in y for every fixed x by construction, see (4.4), (2.9)–(2.10). Consequently, for every $x \in \mathbb{R}_* \setminus (E_\varphi \cup \partial_{\mathbb{R}} Z_\varphi)$, the sum of the derivatives of V in the directions of inner normals (i.e., the sum occurring in (4.1)) coincides in our case with

$$(4.10) \quad \begin{aligned} & 2 \lim_{\substack{y \rightarrow 0 \\ y > 0}} \frac{1}{y} \left(V^\varphi(x + iy) - \varphi(x) \right) \\ &= 2 \lim_{\substack{y \rightarrow 0 \\ y > 0}} \frac{1}{y} \left(\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y}{(t-x)^2 + y^2} \varphi(t) dt - \varphi(x) \right) \\ &= \frac{2}{\pi} \lim_{\substack{y \rightarrow 0 \\ y > 0}} \int_{-\infty}^{+\infty} \frac{\varphi(t) - \varphi(x)}{(t-x)^2 + y^2} dt = \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{\varphi(t) - \varphi(x)}{(t-x)^2} dt. \end{aligned}$$

By the conjugate positivity condition (see (2.6)), the right-hand side in (4.10) is positive for every x . Consequently, condition (4.1) in Proposition 1 is fulfilled, and the function V^φ is subharmonic in $\mathbb{C}_* \setminus (E_\varphi \cup \partial_{\mathbb{R}} Z_\varphi)$. Putting $A_0 := E_\varphi \cup \partial_{\mathbb{R}} Z_\varphi$, $E_0 := E_\varphi$, $Z_0 := \partial_{\mathbb{R}} Z_\varphi$ and applying the final part of Proposition 1, we see that V^φ is subharmonic on the open set $\mathcal{O}_0 := \mathbb{C}_* \supset A_0$. Thus, $V^\varphi \in \text{sbh}(\mathbb{C}_*)$. As a byproduct, identities (4.10) prove that (4.1) is fulfilled and, therefore, also (4.7), but without $(-H\varphi)'$ for the moment.

We pass to the verification of the seminormalization condition 3) in Definition 2. The seminormalization condition at zero (see (2.5)) implies that, for an arbitrary $\varepsilon > 0$, for some $\alpha \in (0, 1]$ depending only on ε and φ we have

$$(4.11) \quad \varphi(x) \leq (1 + \varepsilon) \log \frac{1}{|x|}, \quad |x| \leq \alpha.$$

Moreover, by the compact support condition (2.4), in which we may assume that $R_\varphi \geq 1$, there exists a constant $C_1 \geq 0$ such that

$$(4.12) \quad \varphi(x) \leq C_1, \quad |x| \geq \alpha.$$

Taking the finite support condition into account and coarsening (4.11) and (4.12) slightly, we may unite these two estimates:

$$(4.13) \quad \varphi(x) \leq (1 + \varepsilon) \log^+ \frac{e^{C_1 R_\varphi}}{|x|} =: (1 + \varepsilon) \log^+ \frac{C}{|x|}, \quad C := e^{C_1 R_\varphi}, \quad x \in \mathbb{R}_*.$$

Since the Poisson kernel in (2.9) is positive, the last inequality implies

$$(4.14) \quad V^\varphi(\zeta) = (\mathbb{P}_{\mathbb{C}_\pm} \varphi)(\zeta) \leq (1 + \varepsilon) \left(\mathbb{P}_{\mathbb{C}_\pm} \log^+ \frac{C}{|\cdot|} \right)(\zeta), \quad \zeta \in \mathbb{C}_*.$$

It can easily be checked that for some constant C_2 depending only on C , we have

$$\left(\mathbb{P}_{\mathbb{C}_\pm} \log^+ \frac{C}{|\cdot|} \right)(\zeta) \leq \log \frac{1}{|\zeta|} + C_2, \quad \zeta \in \mathbb{D}_*.$$

By (4.14), it follows that

$$V^\varphi(\zeta) := (\mathbb{P}_{\mathbb{C}_\pm} \varphi)(\zeta) \leq \log \frac{1}{|\zeta|} + C_2, \quad \zeta \in \mathbb{D}_*.$$

Therefore,

$$\limsup_{\substack{\zeta \rightarrow 0, \\ \zeta \in \mathbb{C}_*}} \frac{V^\varphi(\zeta)}{-\log |\zeta|} \leq 1 + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the seminormalization condition (4.5).

We pass to the proof of (4.6). For $\zeta = x + iy$, $|\zeta| = r \geq 2R_\varphi$, we consider separately the case where $|x| \geq y$ and $y > |x|$. It suffices to give a proof for the closed upper half-plane, that is, for $y \geq 0$. If $|x| \geq y$, then for $|t| \leq R_\varphi$ we have

$$(t - x)^2 \geq (|x| - |t|)^2 \geq (r/\sqrt{2} - R_\varphi)^2 \geq (r/\sqrt{2} - r/2)^2 = \left(\frac{\sqrt{2} - 1}{2} \right)^2 r^2.$$

Consequently, if $|x| \geq y$ and $r \geq 2R_\varphi$, then

$$V^\varphi(\zeta) \leq \frac{1}{\pi} \int_{-R_\varphi}^{R_\varphi} \varphi(t) \frac{y}{(t - x)^2} dt \leq \text{const.} \int_{-R_\varphi}^{R_\varphi} \varphi(t) dt \cdot \frac{y}{r^2} \leq \text{const.} \left| \text{Im} \frac{1}{\zeta} \right|.$$

Similarly, for $|x| < y$ and $r \geq 2R_\varphi$ we obtain

$$V^\varphi(\zeta) \leq \frac{1}{\pi} \int_{-R_\varphi}^{R_\varphi} \varphi(t) \frac{y}{y^2} dt \leq \text{const.} \int_{-R_\varphi}^{R_\varphi} \varphi(t) dt \cdot \frac{2y}{r^2} \leq \text{const.} \left| \text{Im} \frac{1}{\zeta} \right|.$$

So, estimate (4.6) is proved.

Finally, to verify the last identity in (4.7), we act as in [27, Chapter XI, §E, 2, Remark]. Consider the following two functions:

$$(\mathbb{P}_{\mathbb{C}_\pm} \varphi)(z), \quad (H\varphi)(z) := \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\text{Re } z - t}{|z - t|^2} + \frac{t}{t^2 + 1} \right) \varphi(t) dt, \quad z \in \mathbb{C}_+.$$

They are harmonic in \mathbb{C}_+ and have continuous second derivatives, maybe, outside an exceptional set $E_\varphi \cup \partial_{\mathbb{R}} Z_\varphi$. By the Cauchy–Riemann equations, we obtain

$$\begin{aligned} (H\varphi)'(x) &= \lim_{y \rightarrow 0^+} \frac{\partial(H\varphi)(x + iy)}{\partial x} \\ &= - \lim_{x \rightarrow 0^+} \frac{\partial(\mathbb{P}_{\mathbb{C}_\pm} \varphi)(x + iy)}{\partial y} = \lim_{y \rightarrow 0^+} \frac{\varphi(x) - (\mathbb{P}_{\mathbb{C}_\pm} \varphi)(x + iy)}{y}, \end{aligned}$$

where the last limit can be written in the form

$$\frac{1}{\pi} \int_0^{+\infty} \frac{2\varphi(x) - \varphi(x + t) - \varphi(x - t)}{t^2} dt.$$

Together with the fact (mentioned in the Introduction) that conditions (2.6) and (2.7) coincide, this gives the identity in (2.6) and the last identity in (4.7). \square

4.2. Restriction of Jensen potentials to $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$. The following statement is a sort of converse to Proposition 2.

Proposition 3. *If V is a Jensen potential continuous on \mathbb{R}_* , and its restriction $\varphi := V|_{\mathbb{R}_*}$ belongs to $C^m(\mathbb{R}_* \setminus (Z_V \cup E_V))$ for some $m \in (\mathbb{N} \setminus \{1\}) \cup \{\infty\}$, where $Z_V := \{x \in \mathbb{R}_* : V(x) = 0\}$ and $E_V \subset \mathbb{R}_*$ is a closed polar subset of \mathbb{C} , then $\varphi \in R\mathcal{P}_0^m$ and*

$$(4.15) \quad V^\varphi(\zeta) := (\mathbb{P}_{\mathbb{C}_\pm} \varphi)(\zeta) \geq V(\zeta), \quad \zeta \neq 0.$$

Proof. The conditions of positivity, compact support (2.4), and seminormalization at zero (2.5) (see the definition of the classes $R\mathcal{P}_0^m$) are fulfilled by the Definition 2 of the Jensen potentials. The function V^φ is continuous on \mathbb{C}_* (see [25, Chapter I, Lemma 3.3]) and is harmonic in \mathbb{C}_\pm . Since the Jensen potential V is subharmonic in \mathbb{C}_* and its restriction to \mathbb{R}_* coincides with the boundary values of the function V^φ on \mathbb{R}_* , it follows that V^φ majorizes V on \mathbb{C}_* , i.e., we have (4.15). For every $x_0 \in \mathbb{R}_*$, the mean value of V^φ on the disk $x_0 + r\mathbb{D}$, $r < |x_0|$, does not exceed the mean value of V on the same disk. Since V is subharmonic, the latter quantity is greater than or equal to $\varphi(x_0) = V^\varphi(x_0)$. Therefore, V^φ is subharmonic in \mathbb{C}_* . Now, the requirement that the inverse Hilbert transform $-H\varphi$ should be monotone increasing on each connected component of the set $\mathbb{R}_* \setminus (Z_\varphi \cup E_\varphi)$ (open in \mathbb{R}_*) can be verified in the form of the conjugate positivity condition (2.6). For this, we use the theory of distributions.

Let $x_0 \in \mathbb{R}_* \setminus (E_\varphi \cup Z_\varphi)$. Choose $r > 0$ so small that

$$(4.16) \quad \Sigma := (x_0 - r, x_0 + r) \subset \mathbb{R}_* \setminus (E_\varphi \cup Z_\varphi).$$

By Lemma B, in the notation (4.3), the corresponding restrictions of V^φ belong to the classes (4.9), and even they are infinitely differentiable on D_+ and D_- , because they are harmonic.

Let $\psi_0 \in C_0^\infty(x_0 - r, x_0 + r) := C_0^\infty((x_0 - r, x_0 + r))$ be a *positive* function. We write the second Green formula in these half-disks for a positive extension $\psi \in C_0^\infty(x_0 + r\mathbb{D})$ of $\psi_0 = \psi|_{(x_0 - r, x_0 + r)}$, and for V^φ :

$$(4.17+) \quad \int_{x_0 - r}^{x_0 + r} \left(V^\varphi(t) \frac{\partial \psi}{\partial \bar{n}_+^{\text{in}}}(t) - \psi(t) \frac{\partial V^\varphi}{\partial \bar{n}_+^{\text{in}}}(t) \right) dt = \iint_{D_+} (\psi \Delta V^\varphi - V^\varphi \Delta \psi) dx dy,$$

$$(4.17-) \quad \int_{x_0 - r}^{x_0 + r} \left(V^\varphi(t) \frac{\partial \psi}{\partial \bar{n}_-^{\text{in}}}(t) - \psi(t) \frac{\partial V^\varphi}{\partial \bar{n}_-^{\text{in}}}(t) \right) dt = \iint_{D_-} (\psi \Delta V^\varphi - V^\varphi \Delta \psi) dx dy.$$

Since ψ is smooth and the normals \bar{n}_+^{in} and \bar{n}_-^{in} are positively oriented, we have

$$(4.18) \quad \frac{\partial \psi}{\partial \bar{n}_+^{\text{in}}}(t) + \frac{\partial \psi}{\partial \bar{n}_-^{\text{in}}}(t) = \frac{\partial \psi}{\partial \bar{n}_+^{\text{in}}}(t) + \frac{\partial \psi}{\partial (-\bar{n}_+^{\text{in}})}(t) \equiv 0, \quad t \in (x_0 + r, x_0 + r).$$

Since V^φ is harmonic in \mathbb{C}_\pm , we obtain $\Delta V^\varphi(\zeta) \equiv 0$ on $D_+ \cup D_-$. Together with (4.18) and after adding (4.17+) and (4.17-), the last identity yields

$$(4.19) \quad \int_{x_0 - r}^{x_0 + r} \psi_0(t) \left(\frac{\partial V^\varphi}{\partial \bar{n}_+^{\text{in}}}(t) + \frac{\partial V^\varphi}{\partial \bar{n}_-^{\text{in}}}(t) \right) dt = \iint_{x_0 + r\mathbb{D}} V^\varphi \Delta \psi dx dy,$$

where the right-hand side is positive because V^φ is subharmonic in $x_0 + r\mathbb{D}$ (from a distributional point of view). Thus, formula (4.19) implies

$$(4.20) \quad \frac{\partial V^\varphi}{\partial \bar{n}_+^{\text{in}}}(t) + \frac{\partial V^\varphi}{\partial \bar{n}_-^{\text{in}}}(t) \geq 0, \quad t \in \mathbb{R}_* \setminus (E_\varphi \cup Z_\varphi),$$

because $x_0 \in \mathbb{R}_*$, r in (4.16), and a positive function $\psi_0 \in C_0^\infty(x_0 - r, x_0 + r)$ can be taken arbitrarily. But the sum of the normal derivatives in (4.20) was already calculated in (4.10), and it is equal to the Cauchy principal value of the integral in (2.6). Thus, the conjugate positivity condition (2.6) holds true, and φ belongs to the test class RP_0^m . \square

Examples. Proposition 3 provides the simplest examples of test functions in the class RP_0^∞ , obtained by restriction to \mathbb{R}_* of known Jensen potentials or Green functions for domains $D \ni 0$ with pole at zero, because such functions are a particular case of Jensen potentials. Here are some of them, taken from [18, Example 1] (as usual, $f^+ = \max\{0, f\}$):

- 1) $\varphi(t) := \log^+ \frac{R}{|t|}$, $t \in \mathbb{R}_*$, the restriction of a Jensen potential, namely, of the extended Green function for the disk $R\mathbb{D}$ with pole at zero: $z \mapsto \log^+ \frac{R}{|z|}$, $z \in \mathbb{C}_*$ (see [1, §4, Example 1]);
- 2) $\varphi(t) = \log^+ \frac{R}{|t|} - \frac{1}{2} \left(1 - \frac{t^2}{R^2}\right)^+$, $t \in \mathbb{R}_*$ (see [1, §4, Example 1]);
- 3) the restriction to \mathbb{R}^* of the positive Green function with pole at zero for a bounded domain D in \mathbb{C} with C^1 -boundary and with $0 \in D$ (see [1, §4, Example 3]).

Explicit test functions of more complicated structure that belong to RP_0^∞ will emerge in the sequel in the uniqueness theorems from §6.

§5. PROOFS OF THE MAIN THEOREMS

5.1. Proof of Theorem 1 on the uniqueness sequences for a Bernstein space.

1. Proof of the implications 1) \implies 2), 3). Suppose that the quantities 2) in (2.11) or 3) in Theorem 1 are finite. Then the functions $\varphi \in RP_0^m$ that, in the form $\varphi|_{\mathbb{R}_*}$, are associated with arbitrary Jensen potentials V (see Proposition 3) with the estimate (4.15) are such that the quantity (3.5) in Theorem 3 (occurring in 2)) is finite, because (4.15) holds true for every λ_k , and φ and V coincide on the real axis by construction. By Theorem 3, the finiteness of the lower upper bound (3.5) implies that Λ is a *nonuniqueness* sequence for the Bernstein space B_σ^∞ . This proves the implications 1) \implies 2) and 1) \implies 3) of Theorem 1.

2. Proof of the implications 2), 3) \implies 1) in Theorem 1. Suppose now that Λ is a *nonuniqueness* sequence for the Bernstein space B_σ^∞ . Then, by the equivalence 1) \iff 3) in Theorem 3, there exists a constant C such that for all Jensen potentials $V \in \mathcal{P}_{J_0}$ we have

$$(5.1) \quad \sum V(\lambda_k) \leq \frac{\sigma}{\pi} \int_{-\infty}^{+\infty} V(t) dt + C,$$

where the constant C is independent of $V \in \mathcal{P}_{J_0}$.

Now, we choose an arbitrary test function $\varphi \in RP_0^m$ and, as in (4.4), consider the Poisson transform

$$(5.2) \quad V^\varphi(\zeta) := (\mathbb{P}_{\mathbb{C}_\pm} \varphi)(\zeta), \quad \zeta \in \mathbb{C}_*.$$

By Proposition 2, this function V^φ possesses nearly all properties of a Jensen potential. Specifically, it is subharmonic and positive outside zero, is seminormalized at zero (see (4.5)), and obeys the conjugate positivity condition (2.6)–(2.7). The only possible exception is the compact support condition (see (4.6)), which is replaced by the weaker condition (4.6):

$$V^\varphi(\zeta) \leq b \left| \operatorname{Im} \frac{1}{\zeta} \right|, \quad |\zeta| \geq r, \quad b, r \geq 0 \text{ are constants.}$$

It is quite easy to convert such a function into a Jensen potential: it suffices to consider the function

$$V_\varepsilon^\varphi(\zeta) := (V^\varphi(\zeta) - \varepsilon)^+$$

with arbitrary $\varepsilon > 0$, which already has compact support but has also all the previous properties (that is, this is a true Jensen potential continuous outside zero). As was mentioned in (5.1), such potentials satisfy the inequalities

$$(5.3) \quad \sum_k V_\varepsilon^\varphi(\lambda_k) \leq \frac{\sigma}{\pi} \int_{-\infty}^{+\infty} V_\varepsilon^\varphi(t) dt + C = \frac{\sigma}{\pi} \int_{-\infty}^{+\infty} (\varphi(t) - \varepsilon)^+ dt + C,$$

where the constant C does not depend on a Jensen potential $V_\varepsilon^\varphi \in \mathcal{P}_{J_0}$ (in particular, on the number ε). Letting $\varepsilon > 0$ tend to zero and using (5.2), we obtain

$$\sum_k (\mathbb{P}_{C_\pm} \varphi)(\lambda_k) \leq \frac{\sigma}{\pi} \int_{-\infty}^{+\infty} \varphi(t) dt + C,$$

and the finiteness of the suprema in 2), (2.11), and 3) follows. This proves the implications 2), 3) \implies 1) in Theorem 1.

5.2. Proof of Theorem 2 on the completeness of systems of exponentials in spaces on an interval.

1. Conditions of completeness. The sequences of exponents for collections of exponentials complete in a function space on an interval are in a well-known relationship with uniqueness sequences in the dual space realized as a special space of entire functions of exponential type (see [2], [18, Theorem 2.1.1]). This relationship shows that if Λ is a uniqueness sequence for B_σ^∞ , then the system is complete in any space among $C_{\mathbb{C}}(\bar{I}_{2\sigma})$ and $L^p(I_{2\sigma})$, $p \geq 1$, because the duals to these spaces, when represented as spaces of entire functions, are all included in the Bernstein space B_σ^∞ .

2. Noncompleteness and estimates for the completeness excess. Suppose now that the suprema in (2.11) or in (2.12) (under the restriction $\Lambda \subset \mathbb{R}$) are finite. Then by Theorem 1, Λ is a nonuniqueness sequence for B_σ^∞ . At this point, a case-by-case analysis is required.

The case of the spaces $C_{\mathbb{R}}(\bar{I}_{2\sigma})$ and $L^p(I_{2\sigma})$ for $p \geq 2$. Conversely, let now the suprema in (2.11) or (under the assumption $\Lambda \subset \mathbb{R}$) in (2.12) be bounded above. Then Λ is a nonuniqueness sequence for B_σ^∞ , and there exists a *nonzero* entire function f satisfying (1.4), bounded on the real axis, and vanishing on Λ . There is no loss of generality in assuming that Λ is nonempty and there exists a point $\lambda' \in \Lambda$.

Then the nonzero entire function $F(z) := f(z)/(z - \lambda')$ of exponential type at most σ vanishes on the sequence $\Lambda' := \Lambda \setminus \lambda'$ and, at the same time, belongs to $L_{\mathbb{C}}^2(\mathbb{R})$. By the classical Paley–Wiener theorem, it follows that there exists a function $g \in L^2(-\sigma, \sigma)$ such that

$$(5.4) \quad F(z) = \int_{-\sigma}^{\sigma} e^{izt} g(t) dt, \quad F(\Lambda') = 0.$$

By the well-known duality (see [18, Theorem 2.1.1]), this means that the system $\text{Exp}^{\Lambda'}$ is not complete in $L^2(-\sigma, \sigma)$. But the condition $g \in L^2(-\sigma, \sigma)$ (for $\sigma < \infty$) shows that, for every $1 \leq q < 2$, we still have $g \in L^q(-\sigma, \sigma)$ and, moreover, g is a density of a Borel measure. By the same duality, the system $\text{Exp}^{\Lambda'}$ is not complete in any space among $L^p(I_{2\sigma})$ for $p \geq 2$, $1/p + 1/q = 1$, and $C_{\mathbb{R}}(\bar{I}_{2\sigma})$. The removal of one exponent to attain noncompleteness means precisely (see [18, 1.1.3]) that $\text{exc } \Lambda \leq 0$.

The case of the spaces $L^p(I_{2\sigma})$ for $1 \leq p < 2$. If a sequence $\Lambda \subset \mathbb{C}$, $0 \notin \Lambda$, does not obey condition (2.11) or, under the restriction $\Lambda \subset \mathbb{R}$, it does not obey condition (2.12), then by Theorem 1, Λ is still a nonuniqueness sequence for B_σ^∞ , and there exists a *nonzero* entire function f that satisfies (1.4), is bounded on the real axis, and vanishes on Λ . There is no loss of generality in assuming that Λ contains at least two (possibly, multiple) points $\lambda', \lambda'' \in \Lambda$, because a system of two exponentials is not complete in any of the spaces in question. Now, consider the function

$$F(z) := \frac{f(z)}{(z - \lambda')(z - \lambda'')},$$

which, clearly, is a nonzero entire function of exponential type at most σ , and it vanishes on $\Lambda'' := \Lambda \setminus \{\lambda', \lambda''\}$. Furthermore, $F \in L^p(\mathbb{R})$, $1 \leq p \leq 2$, and by the Hausdorff–Young theorem there exists a function $g \in L^q(-\sigma, \sigma)$, $1/p + 1/q = 1$ (see the generalization of the Paley–Wiener Theorem in [7, 1.7] or [8, 1.7]) such that (compare with (5.4)) we have

$$F(z) = \int_{-\sigma}^{\sigma} e^{izt} g(t) dt, \quad F(\Lambda'') = 0.$$

By the well-known duality [18, Theorem 2.1.1], this means that the system $\text{Exp}^{\Lambda''}$ is not complete in $L^p(-\sigma, \sigma)$, where $p \leq 2$. Removal of two exponentials for attaining noncompleteness means precisely (see [18, 1.1.3]) that $\text{exc } \Lambda \leq 1$.

Note that, under conditions (2.11) or (2.12), the test classes $R\mathcal{P}_0^m$ can always be replaced with the narrower classes of item 3) in Theorem 1.

§6. SOME INITIAL APPLICATIONS

Our Theorems 1 and 2 imply a number of results about (non)uniqueness sequences for Bernstein spaces and about the completeness of systems of exponentials in $C(\bar{I}_d)$ and $L^p(I_d)$ (for instance, the celebrated Beurling–Malliavin theorem on the radius of completeness stated in Subsection 1.2, but the new proof of it is neither shorter, nor simpler than an earlier proof presented in [28], see also the exposition in [29, Chapter III]). So, here we restrict ourselves to simplest consequences of Theorems 1 and 2, which illustrate the specific features of the new approach. Proofs of many other known results from the monographs [5, 1, 7, 8, 9], their generalizations, and, as far as possible, new facts will be exposed elsewhere.

6.1. Uniqueness theorems. It is most easy to apply Theorems 1 and 2 for obtaining conditions sufficient for uniqueness in Bernstein spaces and conditions for completeness of systems of exponentials in $C(\bar{I}_d)$ and $L^p(I_d)$. First, we note that in the proofs of Theorems 1 and 2, as well as in their predecessors Theorem A and Theorem 3 (see [1]), no conditions were imposed except the absence of a limit point for the sequence in question (or the sequence of exponents) and the nonessential requirement that the sequence should not contain zero (see Remark 2). Below we show that all sufficient conditions for uniqueness from the Cartwright theorem (see Subsection 1.2) and the commentaries after it are contained in Theorem 1. In fact, all of them are covered by the following simple consequence of Theorems 1 and 2.

Corollary 1. *Let $\varphi \in R\mathcal{P}_0^m$ be a nonzero function, and let $\sigma > 0$. If for a sequence (1.1) we have*

$$(6.1) \quad \limsup_{r \rightarrow +\infty} \left(\sum_k (P_{\mathbb{C}_\pm} \varphi) \left(\frac{\lambda_k}{r} \right) - r \frac{\sigma}{\pi} \int_{\mathbb{R}} \varphi(x) dx \right) = +\infty,$$

then Λ is a uniqueness sequence for B_σ^∞ , and the system of exponentials Exp^Λ is complete in any of the spaces $C(\bar{I}_{2\sigma})$ and $L^p(I_{2\sigma})$, $p \geq 1$.

The proof of this corollary readily follows from Theorems 1 and 2 if we use the invariance of the class $R\mathcal{P}_0^m$ under homothety, as was mentioned after Definition 1 in (2.8), and make the change of variables $x = t/r$ in the Poisson integrals and the last integral.

Examples. We apply Corollary 1 to certain specific functions φ in the class $R\mathcal{P}_0^m$.

1. Let φ be the function in Example 1) in Subsection 4.2 with $R = 1$, i.e., $\varphi(t) = \log^+ \frac{1}{|t|}$.

Then condition (6.1) for being a uniqueness sequence can be written in the form

$$(6.2) \quad \limsup_{r \rightarrow +\infty} \left(\sum_{\operatorname{Im} \lambda_k \neq 0} \frac{1}{\pi} \int_{-1}^1 \left| \operatorname{Im} \frac{1}{x - \lambda_k/r} \right| \log \frac{1}{|x|} dx + \sum_{\operatorname{Im} \lambda_k = 0} \log^+ \frac{|\lambda_k|}{r} - \frac{2\sigma}{\pi} r \right) = +\infty.$$

In particular, the first sum here is majorized by

$$\sum_{\operatorname{Im} \lambda_k \neq 0} \log^+ \frac{|\lambda_k|}{r}$$

(see Proposition 3, inequality (4.15)). Thus, (6.2) implies also the weaker condition that guarantees uniqueness, namely,

$$(6.3) \quad \limsup_{r \rightarrow +\infty} \left(\sum_k \log^+ \frac{|\lambda_k|}{r} - \frac{2\sigma}{\pi} r \right) \stackrel{(1.8)}{=} \limsup_{r \rightarrow +\infty} \left(N_\Lambda(r) - \frac{2\sigma}{\pi} r \right) = +\infty,$$

which is in full accordance with the Levinson theorem (see Subsection 1.2) with $p = +\infty$, but, surely, does not prove it completely for $p < +\infty$. In particular, this implies some elementary facts: if Λ is a nonuniqueness sequence for B_σ^∞ , or $\operatorname{Exp}^\Lambda$ is not complete in $C(\bar{I}_d)$ or $L^p(I_d)$, then Λ must have finite averaged upper density; consequently, it must be of finite upper density because $n_\Lambda(t) \leq N_\Lambda(et)$ for all $t > 0$, the convergence exponent does not exceed 1 and, *a fortiori*, the series (1.6) converges.

At the same time, condition (6.3) is indeed weaker than (6.2), because now we can deduce the Cartwright theorem (see Subsection 1.2) from it. Suppose that Λ is a nonuniqueness sequence for some B_σ^∞ . Then the first sum in (6.2) is finite, for example, for $r = 1$. Summation and integration can be interchanged in this case, because all summands and the function φ are positive, see [16, Chapter IV, §4, Corollary 4]. Therefore, the series (1.5) converges for at least one $t \in (-1, 1)$, the convergence of the series (1.6) implies that of the series (1.5) for every t , and the Cartwright theorem follows.

2. Let φ be the function in Example 2) in Subsection 4.2 with $R = 1$, i.e.,

$$\varphi(t) = \log^+ \frac{1}{|t|} - \frac{1}{2}(1 - t^2)^+, \quad t \in \mathbb{R}_*.$$

Then condition (6.1) for uniqueness sequences can be rewritten in the form

$$(6.4) \quad \limsup_{r \rightarrow +\infty} \left(\sum_{\operatorname{Im} \lambda_k \neq 0} \frac{1}{\pi} \int_{-1}^1 \left| \operatorname{Im} \frac{1}{x - \lambda_k/r} \right| \left(\log \frac{1}{|x|} - \frac{1}{2}(1 - x^2) \right) dx + \sum_{\operatorname{Im} \lambda_k = 0} \left(\log^+ \frac{r}{|\lambda_k|} - \frac{1}{2} \left(1 - \frac{|\lambda_k|^2}{r^2} \right)^+ \right) - \frac{4\sigma}{3\pi} r \right) = +\infty.$$

In particular, the first sum here is at least

$$\sum_{\operatorname{Im} \lambda_k \neq 0} \left(\log^+ \frac{r}{|\lambda_k|} - \frac{1}{2} \left(1 - \frac{|\lambda_k|^2}{r^2} \right)^+ \right)$$

(see Proposition 3, inequality (4.15)). Thus, formula (6.4) implies the following condition for being a uniqueness sequence:

$$(6.5) \quad \limsup_{r \rightarrow +\infty} \left(\sum_{|\lambda_k| \leq r} \left(\log \frac{r}{|\lambda_k|} - \frac{1}{2} \left(1 - \frac{|\lambda_k|^2}{r^2} \right) \right) - \frac{4\sigma}{3\pi} r \right) = +\infty.$$

Representing the sum here as a Stiltjes integral, after a series of integrations by parts we see that condition (6.5) for nonuniqueness sequences coincides with the relation

$$(6.6) \quad \limsup_{r \rightarrow +\infty} \left(\frac{1}{r^2} \int_0^r N_\Lambda(t) t dt - \frac{4\sigma}{3\pi} r \right) = +\infty.$$

It can easily be shown that if the last condition is fulfilled, then (6.3) holds true, i.e., we obtain nothing new in (6.6) compared to (6.3). However, we do not know if the same happens with condition (6.4).

3. We show an “asymmetric” example of application of Corollary 1.

Let $\zeta = te^{i\theta} \in \mathbb{C}$, and let k be a γ -trigonometrically convex (see⁶ [2]) continuous 2π -periodic positive function. Throughout this item, we assume that $\max k \leq 1$. For $a > 1$ and $t > 1$, we put

$$(6.7) \quad V_k(\zeta; a) = \frac{a^\gamma}{2\gamma(a^{2\gamma} + 1)} k(\theta) \left(\left(\frac{a}{t} \right)^\gamma - \left(\frac{t}{a} \right)^\gamma \right)^+,$$

and for $0 < t < 1$ we put

$$(6.8) \quad V_k(\zeta; a) = \frac{a^\gamma}{2\gamma(a^{2\gamma} + 1)} k(\theta) \left((at)^\gamma - \left(\frac{1}{at} \right)^\gamma \right)^+ + \log^+ \frac{1}{t}.$$

In [18, §7] it was shown that the functions defined above are continuous Jensen potentials smooth outside the points $t = a$ and $t = 1$. Accordingly, by Proposition 3, their restrictions to \mathbb{R}_* belong to the class RP_0^∞ of test functions. Here we restrict ourselves to the simplest example of the function

$$(6.9) \quad k(\theta) = \frac{1}{2} |1 + \cos \theta|, \quad V_k := V_k(\cdot; a).$$

In this case it is easy to check that the function (6.9) is γ -trigonometrically convex for every $\gamma \geq 1/2$. The replacement of the Poisson integrals in items 2–3 of Theorem 1 at the points λ_k by the Jensen potentials themselves will only lead to smaller quantities (see Proposition 3, inequality 4.15). So, in the case under study, in Corollary 1 we can choose the functions (6.7)–(6.8) themselves in place of the Poisson integrals.

Let $\gamma \geq 1/2$. For simplicity, suppose that $\gamma < 1$. If the function k is chosen as in (6.9), the function V is chosen as in (6.7)–(6.8), and

$$\limsup_{r \rightarrow +\infty} \left(\sum_k V(\lambda_k/r; a) - r \frac{\sigma}{\pi} \left((a-1/a) \frac{2\gamma}{1-\gamma^2} - \frac{1}{1-\gamma} (a^\gamma - a^{-\gamma}) + \frac{1}{1+\gamma} (a^{-\gamma} - a) + \frac{2\gamma(a^{2\gamma} + 1)}{a^\gamma} \right) \right) = +\infty,$$

then Λ is a uniqueness sequence for B_σ^∞ .

The cases where $\gamma = 1$ and $\gamma > 1$ are much subtler, and we do not consider them here (cf. [18, §7, Uniqueness theorem, items 2)–3]).

We present a short proof of the Schwartz theorem stated in the Introduction.

⁶The main condition is that $k'' + \gamma^2 k \geq 0$ in the sense of the theory of distributions.

Proof of the Schwartz theorem. Under condition (1.9), there is no loss of generality in assuming that for sufficiently small $\varepsilon > 0$ we have

$$(6.10) \quad \sum_{k \in \mathbb{N}} \left| \operatorname{Im} \frac{1}{\lambda_k} \right| < \varepsilon$$

in place of the second condition in (1.9). To ensure this, it suffices to shift a finite number of points in Λ ; this does not affect (non)completeness, see Remark 2. The first estimate in (1.9) implies $|\operatorname{Im} \lambda_k| \geq |\lambda_k| \sin \alpha$. This yields an estimate for the Poisson kernel at every point λ_k :

$$\frac{1}{\pi} \frac{|\operatorname{Im} \lambda_k|}{(t - \operatorname{Re} \lambda_k)^2 + (\operatorname{Im} \lambda_k)^2} \leq \frac{1}{\pi \sin^2 \alpha} \frac{|\operatorname{Im} \lambda_k|}{|\lambda_k|^2} = \frac{1}{\pi \sin^2 \alpha} \left| \operatorname{Im} \frac{1}{\lambda_k} \right|.$$

By (6.10), it follows that for every test function $\varphi \in R\mathcal{P}_0^m$ we have

$$\begin{aligned} \sum_{k \in \mathbb{N}} (\mathbb{P}_{\mathbb{C}^\pm} \varphi)(\lambda_k) &:= \sum_{k \in \mathbb{N}} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{|\operatorname{Im} \lambda_k|}{(t - \operatorname{Re} \lambda_k)^2 + (\operatorname{Im} \lambda_k)^2} \varphi(t) dt \\ &\leq \sum_{k \in \mathbb{N}} \frac{1}{\pi \sin^2 \alpha} \left| \operatorname{Im} \frac{1}{\lambda_k} \right| \cdot \int_{-\infty}^{+\infty} \varphi(t) dt \leq \frac{\varepsilon}{\pi \sin^2 \alpha} \cdot \int_{-\infty}^{+\infty} \varphi(x) dx. \end{aligned}$$

If ε is chosen sufficiently small, then $\frac{\varepsilon}{\pi \sin^2 \alpha} \leq \frac{2\sigma'}{2\pi} < \frac{2\sigma}{2\pi}$, and the supremum in (2.11) (or in (2.12)) is finite. By the second part of the theorem on completeness, the system $\operatorname{Exp}^\Lambda$ is not complete in $C(I_{2\sigma'})$ and in $L^p(I_{2\sigma'})$ after removal of one or two exponentials. Thus, the system $\operatorname{Exp}^\Lambda$ is not complete in $C(I_d)$ and in $L^p(I_d)$ for every $d > 0$. \square

6.2. Stability for completeness and subsequences of zeros. Apparently, the following result is new.

Theorem 4. *Let $\Lambda = \{\lambda_k\}_{k \in \mathbb{N}}$ and $\Gamma = \{\gamma_k\}_{k \in \mathbb{N}}$ be two sequences of points in $\mathbb{C} \setminus \mathbb{R}$ without limit points in \mathbb{C} . Suppose that the sequences have finite upper density and that the sums (cf. (1.5))*

$$(6.11) \quad \sum_k \left| \operatorname{Im} \frac{1}{\lambda_k} \right|, \quad \sum_k \left| \operatorname{Im} \frac{1}{\gamma_k} \right|$$

are finite. Let Λ be a subsequence of zeros for B_σ^∞ . If for some $k_0 \in \mathbb{N}$ and $s \in \mathbb{R}$ we have

$$(6.12) \quad \sup_{t \in \mathbb{R}} \left(\sum_{k > k_0} \left| \operatorname{Im} \frac{1}{t - \gamma_k} \right| - \sum_{k > k_0} \left| \operatorname{Im} \frac{1}{t - \lambda_k} \right| \right) \leq s,$$

then $s > -\sigma$ and Γ is a subsequence of zeros for $B_{\sigma+s}^\infty$. In particular, if $s = 0$, then Γ is a subsequence of zeros for the same space B_σ^∞ .

Together with Theorem 2, this implies the following corollary in an easy and standard way.

Corollary 2. *Suppose that the assumptions of Theorem 4 are fulfilled, up to the finiteness of the sums (6.11) inclusive. If (6.12) is fulfilled for some $s \in \mathbb{R}$ and $2s < d$, then the excess $\operatorname{exc} \Lambda$ for $C(\bar{I}_{d-2s})$ or $L^p(I_{d-2s})$ with $p \geq 2$ cannot be smaller than $\operatorname{exc} \Gamma - 1$, where $\operatorname{exc} \Gamma$ is the excess for $C(\bar{I}_d)$, respectively, $L^p(I_d)$ with the same value $p \geq 2$; also, the quantity $\operatorname{exc} \Lambda$ for $L^p(I_{d-2s})$ with $1 \leq p < 2$ cannot be smaller than $\operatorname{exc} \Gamma - 2$ for $L^p(I_d)$ with the same value of p .*

Some commentary to this corollary is probably in order in the case where $2s \geq d$. Then either $d = 2s$, $\bar{I}_{d-2s} = \bar{I}_0$ is a singleton, and an arbitrary system of exponentials (even that consisting of only one function) is complete in $C(\bar{I}_0)$, and $I_{d-2s} = I_0 = \emptyset$. Also for $d < 2s$, we have $I_{d-2s} = \bar{I}_{d-2s} = \emptyset$, and function spaces on an empty set are empty.

Proof of Theorem 4. Since Λ is a subsequence of zeros for B_σ^∞ and $\operatorname{Im} \lambda_k \neq 0$ for all k , by Theorem 1 and Remark 4 we can find a constant $C \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} \left(\sum_k \frac{1}{\pi} \left| \operatorname{Im} \frac{1}{t - \lambda_k} \right| \right) \varphi(t) dt - \frac{\sigma}{\pi} \int_{\mathbb{R}} \varphi(t) dt \leq C, \quad \varphi \in R\mathcal{P}_0^m.$$

Therefore, by (6.12), we have

$$(6.13) \quad \int_{\mathbb{R}} \left(\sum_k \frac{1}{\pi} \left| \operatorname{Im} \frac{1}{t - \gamma_k} \right| \right) \varphi(t) dt - \frac{\sigma + s}{\pi} \int_{\mathbb{R}} \varphi(t) dt \leq C, \quad \varphi \in R\mathcal{P}_0^m.$$

If $s \leq -\sigma$, then, denoting by γ_1 some point of Γ , from the last statement and the positivity of φ we deduce that

$$(6.14) \quad \frac{1}{\pi} \int_{\mathbb{R}} \left| \operatorname{Im} \frac{1}{t - \gamma_1} \right| \varphi(t) dt \leq C, \quad \varphi \in R\mathcal{P}_0^m.$$

But the left-hand side here cannot be uniformly bounded if, for instance, the functions φ run over the class (see examples in Subsection 4.2 above) of the functions $t \mapsto \log^+(R/|t|)$, $t \in \mathbb{R}_*$, for $0 < R \rightarrow +\infty$. Indeed, in this case the left-hand side of (6.14) cannot be smaller than

$$\frac{1}{\pi} \int_{\mathbb{R}} \left| \operatorname{Im} \frac{1}{t - \gamma_1} \right| \log \frac{R}{|t|} dt = \log \frac{R}{|\gamma_1|} \xrightarrow{R \rightarrow +\infty} +\infty.$$

Thus, $s > -\sigma$, i.e., $\sigma + s > 0$. Now, (6.13) and the implication 1) \Rightarrow 2) in Theorem 1 show that Γ is a nonuniqueness sequence, or a subsequence of zeros for the Bernstein space $B_{\sigma+s}^\infty$. \square

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