

Zero (sub)sets for spaces of holomorphic functions and (sub)harmonic minorants

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Abstract

Let $\Lambda = \{\lambda_k\}$ be a sequence of points in a domain Ω of the complex plane \mathbb{C} . We obtain various general conditions in terms of the balayage and Green's functions under which the sequence Λ is the zero (sub)set for weighted spaces of holomorphic functions in Ω . In particular, we consider the classical space A^{-p} with $p > 0$, i. e., the set of functions f holomorphic in the unit disk \mathbb{D} satisfying $\sup_{z \in \mathbb{D}} |f(z)|(1-|z|)^p < +\infty$.

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Introduction

Let Ω be a domain in the complex plane \mathbb{C} with the boundary $\partial\Omega$. Denote by $\text{Hol}(\Omega)$ the space of all holomorphic functions in Ω . We are concerned with finite or infinite sequences $\Lambda = \{\lambda_k\}$, $k = 1, 2, \dots$ of not necessarily distinct points from the domain Ω , without limit points in Ω . Let n_Λ be an integer-valued *counting measure* of sequence Λ , defined by

$$n_\Lambda(S) := \sum_{\lambda_k \in S} 1, \quad S \subset \Omega. \quad (0.1)$$

The function

$$n_\Lambda(z) := n_\Lambda(\{z\}), \quad z \in \Omega, \quad (0.2)$$

is called the *divisor* of sequence Λ . In our paper the sequence Λ coincides with a sequence $\Gamma = \{\gamma_n\}$ (or is equal to Γ , or $\Lambda = \Gamma$) if and only if $n_\Lambda(z) = n_\Gamma(z)$ for every $z \in \Omega$; $\Lambda \subset \Gamma$ means $n_\Lambda(z) \leq n_\Gamma(z)$ for all $z \in \Omega$.

Given $f: A \rightarrow B$ and $b \in B$, we write $f \equiv b$ on A' if f is identically equal to b on $A' \subset A$; in the opposite case, $f \not\equiv b$ on A' . For a subset $A' \subset A$, denote by $f|_{A'}$ the restriction of f to A' .

Let $A, B \subset [-\infty, +\infty]$. A function $f: A \rightarrow B$ is *increasing* (*decreasing* resp.) if, for any $x_1, x_2 \in A$, the inequality $x_1 \leq x_2$ implies the inequality $f(x_1) \leq f(x_2)$ ($f(x_1) \geq f(x_2)$ resp.).

“Positive” (“negative” resp.) means “ ≥ 0 ” (“ ≤ 0 ” resp.).

Let $f \in \text{Hol}(\Omega)$, $f \not\equiv 0$ on Ω . We denote by Zero_f the sequence of zeros, counting multiplicities, of the function f . The sequence Λ is **zero set** for a class $H \subset \text{Hol}(\Omega)$ (we write $\Lambda \in \text{Zero}(H)$) if and only if there exists a function $f \in H$ such that $\Lambda = \text{Zero}_f$. A function $f \in \text{Hol}(\Omega)$ *vanish on* Λ if and only if $\Lambda \subset \text{Zero}_f$ (we write $f(\Lambda) = 0$). The sequence Λ is **zero subset** for a class $H \subset \text{Hol}(\Omega)$ if and only if there exists a nonzero function $f \in H$ which vanish on Λ .

By \mathbb{D} denote the *unit disk* $\{z \in \mathbb{C}: |z| < 1\}$.

By m denote the *Lebesgue measure* on \mathbb{C} .

We write $D \Subset \Omega$ if the closure \overline{D} of $D \subset \Omega$ is a compact subset in Ω , i. e., the set D is a relatively compact subset of Ω .

For domain $D \subset \mathbb{C}$, denote by $g_D(\cdot, z)$ the *extended Green's function* for D with a pole at $z \in D$ [1, 5.7.4], i. e., $g_D(\zeta, z) \equiv 0$ for all $\zeta \in \mathbb{C} \setminus \overline{D}$, and $g(\zeta, z)$ is a subharmonic function of $\zeta \in \mathbb{C} \setminus \{z\}$.

For a real-valued function M on an interval of the real axis \mathbb{R} , denote by M'_- the *left-hand derivative* of M .

Given $M: \Omega \rightarrow [-\infty, +\infty]$, we put

$$\text{Hol}(\Omega; M) := \left\{ f \in \text{Hol}(\Omega) : \sup_{z \in \Omega} \frac{|f(z)|}{\exp M(z)} < +\infty \right\}. \quad (0.3)$$

A function $M: \mathbb{D} \rightarrow \mathbb{R}$ is called *radial* if $M(z) = M(|z|)$ for all $z \in \mathbb{D}$. Given $z \in \mathbb{C}$ and $t \in \mathbb{R}$, denote by $D(z, t)$ the open disk of radius t , centered at z . If $t \leq 0$, then $D(z, t)$ is the *empty set* \emptyset . By definition $D(t) := D(0, t)$.

Hereinafter by Ω denote a domain in \mathbb{C} containing the origin, i. e., $0 \in \Omega$.

By $\mathcal{U}_0^d(\Omega)$ denote the class of all subdomains $D \Subset \Omega$ such that each domain $D \in \mathcal{U}_0^d(\Omega)$ is the union of finite number of disks $D(z, t) \Subset \Omega$, and D contains the origin, i. e., $0 \in D$.

Let $\Omega_0 \neq \emptyset$ be a relatively compact subdomain of Ω , and the domain Ω_0 contains the origin. By $\mathcal{U}_0^d(\Omega; \Omega_0)$ denote the subclass of $\mathcal{U}_0^d(\Omega)$ such that each domain $D \in \mathcal{U}_0^d(\Omega; \Omega_0)$ includes the domain Ω_0 ; $\mathcal{U}_0^d(\Omega; \emptyset) := \mathcal{U}_0^d(\Omega)$.

The following Theorems 0.1 and 0.2 illustrate a some of our results.

Theorem 0.1. *Let $M: \mathbb{D} \rightarrow [0, +\infty)$ be a radial positive function that is continuous at 0. Suppose that $M|_{[0,1]}$ is the increasing convex function of \log on $(0, 1)$, i. e., the superposition $M \circ \exp$ is convex function on $(-\infty, 0)$. Under the condition*

$$\int_r^{1^-} (1-t) dM(t) = O(1-r) \quad \text{as } r \rightarrow 1^-, \quad t, r \geq 0, \quad (0.4)$$

the following three statements hold:

(Z_r) *A sequence $\Lambda = \{\lambda_n\}$, $0 \notin \Lambda$, is **zero set** for the space $\text{Hol}(\mathbb{D}; M)$ if and only if there are constants $a < 1$ and C such that the inequality*

$$\sum_n g_D(\lambda_n, 0) \leq \int_0^{1^-} \left(\frac{1}{2\pi} \int_0^{2\pi} g_D(te^\theta, 0) d\theta \right) d(tM'_-(t)) + C \quad (0.5)$$

holds for each domain $D \in \mathcal{U}_0^d(\mathbb{D}; D(a))$;

(S_r) *A sequence Λ in \mathbb{D} is **zero subset** for the space $\text{Hol}(\mathbb{D}; M)$ if and only if the sequence Λ is a **zero set** for the space $\text{Hol}(\mathbb{D}; M)$;*

(M_r) *If $f = g/q$ is a meromorphic function on \mathbb{D} , and $g, q \in \text{Hol}(\mathbb{D}; M)$, then there are holomorphic functions $g_0, q_0 \in \text{Hol}(\mathbb{D}; M)$ **without common zeros** such that $f = g_0/q_0$ on \mathbb{D} .*

Theorem 0.1 is a very special case of Theorem 1.1 \Leftarrow Corollary 1.1 \Leftarrow the Main Theorem (see exact wordings of these results in Subsection 1.2). Corollary 1.1 is established for spaces $H(\Omega; M)$ in case that Ω is a simply connected domain, and $M \in SH(\Omega)$. In Theorem 1.1 we consider generally speaking a case of non-radial subharmonic function M on \mathbb{D} .

In our article a main model class $\text{Hol}(\mathbb{D}; M)$ is the space A^{-p} , $p \geq 0$, with

$$M(z) \equiv M(|z|) \equiv p \log \frac{1}{1-|z|}, \quad z \in \mathbb{D},$$

i. e., the set of functions $f \in \text{Hol}(\mathbb{D})$ satisfying

$$|f(z)| \leq C_f \left(\frac{1}{1-|z|} \right)^p, \quad \forall z \in \mathbb{D},$$

where C_f is a constant dependent on f . For $p = 0$ this space is the space H^∞ of bounded holomorphic functions in \mathbb{D} . The classical Nevanlinna theorem gives precise geometric information about the zero sets for H^∞ : *a sequence Λ is zero set for H^∞ if and only if $\sum_{\lambda_k \in \mathbb{D}} (1 - |\lambda_k|) < +\infty$ (the classical Blaschke condition), and class of zero sets for H^∞ coincides with the class of*

of zero subsets for H^∞ . B. Korenblum introduced in the work [2] a notion of density, in a certain sense generalizing the classical Blaschke condition, and found a complete geometric description of zero sets for the algebra $A^{-\infty} = \cup_{0 \leq p} A^{-p}$. E. Beller [3] proved that for any $p > 0$, the class of zero sets for A^{-p} coincides with the class of zero subsets for A^{-p} . D. Pascuas [4] and J. Bruna and X. Massaneda [5] generalized the result of B. Korenblum to weighted algebras $A^\lambda(\mathbb{D}) := \{f \in \text{Hol}(\mathbb{D}) : \log |f(z)| \leq C_f \lambda(|z|)\}$ where $\lambda > 0$ is a “slowly increasing” function, and C_f is a constant.

In [6]–[7] K. Seip evolved the method of Korenblum and obtained a similar complete description of zero set for spaces $A_+^{-p} = \cap_{p' > p} A^{-p'}$ and $A_-^{-p} = \cup_{p' < p} A^{-p'}$. The joint book of H. Hedenmalm, B. Korenblum, and K. Zhu [8, Ch. 4] contains a detailed analysis of this results together with their improved interpretations.

D. Luecking gave in [9] a criterion of zero sets for A^{-p} in terms of existence of harmonic majorant for special test function constructed by Λ . A development of Luecking’s method to weighted spaces on \mathbb{D} with “slowly increasing” weight was recently proposed in [10].

Following to [11], we will refer to the spaces A^{-p} as *uniform Bergman spaces*.

In our article we establish necessary and sufficient conditions of zero sets for weighted classes of holomorphic functions and their corollaries for the uniform Bergman spaces in terms of balayage of measures and functions (see Theorems 7.1 and 7.2 in Section 7), and in terms of Green’s functions.

Theorem 0.2. *Let $\Lambda = \{\lambda_k\}$, $k = 1, 2, \dots$, $0 \notin \Lambda$, be a sequence of points in \mathbb{D} and $0 \leq p < +\infty$. The following three statements are equivalent:*

- (i) Λ is a zero set for A^{-p} ;
- (ii) There exist constants $a < 1$ and C such that for any $D \in \mathcal{U}_0^d(\mathbb{D}; D(a))$,

$$\sum_k g_D(\lambda_k, 0) \leq p \int_0^1 \left(\frac{1}{2\pi} \int_0^{2\pi} g_D(te^{i\theta}, 0) d\theta \right) \frac{dt}{(1-t)^2} + C; \quad (0.6)$$

- (iii) There are constants $a < 1$ and C such that for any $D \in \mathcal{U}_0^d(\mathbb{D}; D(a))$,

$$\begin{aligned} \sum_k (1 - |\lambda_k|^2)^2 \left(\frac{1}{\pi} \int_{\mathbb{D}} g_D(\zeta, 0) \frac{dm(\zeta)}{|1 - \lambda_k \bar{\zeta}|^4} \right) \\ \leq p \int_0^1 \left(\frac{1}{2\pi} \int_0^{2\pi} g_D(te^{i\theta}, 0) d\theta \right) \frac{dt}{(1-t)^2} + C. \end{aligned} \quad (0.7)$$

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1 Main notions and results

Let Ω be a domain in \mathbb{C} , $0 \in \Omega$, and let S be a subset in Ω . We write $C(S)$ for the space of all continuous real-valued functions on S .

By $\mathcal{M}(S)$ denote the set of all real-valued Borel (Radon) measures on S (on $C(S)$); by $\mathcal{M}^+(S)$ denote the subcone of $\mathcal{M}(S)$ consisting of positive Borel measures; by $\mathcal{M}_{ac}^+(S)$ denote the subset of $\mathcal{M}^+(S)$ consisting of measures that are absolutely continuous with respect to Lebesgue measure m .

Let $\mu \in \mathcal{M}(\Omega)$. Denote by $\text{supp } \mu$ the *support* of μ . We say that a measure $\mu \in \mathcal{M}^+(\Omega)$ is *concentrated* in a subset $S \subset \Omega$ if $\mu(\Omega \setminus S) = 0$. Given Borel set $B \subset S$, we denote by $\mu|_B$ the *restriction of μ to B* .

Given $\nu \in \mathcal{M}(\Omega)$, $z \in \Omega$, $t \geq 0$, we write

$$\nu(z, t) := \nu(D(z, t)) \text{ if } D(z, t) \subset \Omega, \quad \nu^{\text{rad}}(t) := \nu(0, t). \quad (1.1)$$

We hope that last notation will not create confusion with notion (0.2) for the divisor of sequence Λ , defined by the counting measure (0.1), and with notation $d\nu(\zeta)$ which means that the variable of integration is ζ .

By $\text{Harm}(\Omega)$ denote the space of all harmonic functions on Ω , and by $SH(\Omega)$ denote the cone of all subharmonic functions on Ω . The function $\equiv -\infty$ on Ω belongs to $SH(\Omega)$. Besides, $-SH(\Omega)$ is the cone of all superharmonic functions on Ω . For $u \in SH(\Omega)$, $u \not\equiv -\infty$, we denote by¹ $\nu_u := \frac{1}{2\pi} \Delta u$ the *Riesz measure* of u .

1.1 Subharmonic kernels

Let B be a Borel subset of Ω , and $\nu \in \mathcal{M}^+(\Omega)$. By definition, $L^1(B, d\nu)$ is the set of all functions $q: B \rightarrow [-\infty, +\infty]$ that are integrable with respect to the restriction of ν to B , i. e., $\int_B |q| d\nu < +\infty$.

Definition 1. Let B be a Borel subset of Ω . Let

$$h: (\zeta, z) \longrightarrow \mathbb{R}, \quad (\zeta, z) \in B \times \Omega,$$

be a Borel-measurable function which is locally bounded. Suppose, for every fixed point $\zeta \in B$, the function $h(\zeta, \cdot)$ is *harmonic on Ω* ; then the function

$$k(\zeta, z) := \log |\zeta - z| + h(\zeta, z), \quad (\zeta, z) \in B \times \Omega, \quad (1.2)$$

is a *subharmonic kernel on $B \times \Omega$* (supported by B with the harmonic component h).

¹Here Δ is the Laplace operator which acts in sense of distribution theory.

Suppose a $\nu \in \mathcal{M}^+(\Omega)$ is *concentrated in B* , i. e., $\nu(\Omega \setminus B) = 0$. A subharmonic kernel k on $B \times \Omega$ is *suitable for ν* if for any $z \in \Omega$ there are a subdomain $D_z \ni z$ of Ω and a function $q \in L^1((\Omega \setminus D_z) \cap B, d\nu)$ such that

$$|k(\zeta, w)| \leq q(\zeta), \quad \forall \zeta \in (\Omega \setminus D_z) \cap B, \quad \forall w \in D_z.$$

Examples. Consider some frequently occurring subharmonic kernels.

0. The function $\log |\zeta - z|$ is a suitable subharmonic kernel on $\Omega \times \Omega$ for all positive Borel measures with compact support in Ω .
1. If the domain Ω possesses a Green's function g_Ω ($\partial\Omega$ is non-polar), then the functions $-g_\Omega(\zeta, z)$ and $-g_\Omega(\zeta, z) + \log |\zeta|$ are subharmonic kernels on $\Omega \times \Omega$ and on $(\Omega \setminus \{0\}) \times \Omega$ respectively. These kernels are suitable for measures $\nu \in \mathcal{M}^+(\Omega)$ satisfying (see [12, Theorem 4.5.4])

$$\int_{\Omega} g_\Omega(\zeta, 0) d\nu(\zeta) < +\infty.$$

2. Let $\Omega = \mathbb{D}$. Here we use the Blaschke factor, a variant of this factor, and the pseudohyperbolic distance for \mathbb{D} :

$$B_\zeta(z) := \frac{|\zeta|}{\zeta} \frac{\zeta - z}{1 - \bar{\zeta}z}, \quad \zeta \in \mathbb{D} \setminus \{0\}, z \in \mathbb{D}, \text{ but } |B_0(z)| := |z|. \quad (1.3B)$$

$$\bar{B}_\zeta(z) := \frac{\bar{\zeta}(\zeta - z)}{1 - \bar{\zeta}z} = |\zeta| B_\zeta(z), \quad \zeta \in \mathbb{D}, z \in \mathbb{D}, \quad (1.3\bar{B})$$

$$\rho(\zeta, z) := \left| \frac{\zeta - z}{1 - \bar{\zeta}z} \right| = |B_\zeta(z)| = \frac{1}{|\zeta|} |\bar{B}_\zeta(z)|. \quad (1.3\rho)$$

The following functions are subharmonic kernels:

- (B₀) The subharmonic *Blaschke kernel* on $\mathbb{D} \times \mathbb{D}$ is the function $b_1(\zeta, z) := \log |B_\zeta(z)| = \log |\rho(\zeta, z)| = -g_{\mathbb{D}}(\zeta, z)$ which is suitable for measures $\nu \in \mathcal{M}^+(\mathbb{D})$ satisfying $\int_0^{1^-} (1-t) d\nu^{\text{rad}}(t) < +\infty$. Similarly, $\bar{b}_1(\zeta, z) := \log |\bar{B}_\zeta(z)| = -g_{\mathbb{D}}(z, \zeta) + \log |\zeta|$ is a subharmonic kernel on $(\mathbb{D} \setminus \{0\}) \times \mathbb{D}$ which is suitable for the same measures $\nu \in \mathcal{M}^+(\mathbb{D})$.

- (\bar{D}_p) For an integer $p \geq 0$, the subharmonic *Dzhrbashyan's kernel* of

genus² p supported by $\mathbb{D} \setminus \{0\}$ is the function (see [13]–[17])

$$\begin{aligned}\bar{d}_p(\zeta, z) &:= \log|\bar{B}_\zeta(z)| + \sum_{k=1}^p \frac{1}{k} \operatorname{Re}(1 - \bar{B}_\zeta(z))^k \\ &= \log\left|\frac{\bar{\zeta}(\zeta - z)}{1 - \bar{\zeta}z}\right| + \sum_{k=1}^p \frac{1}{k} \operatorname{Re}\left(\frac{1 - |\zeta|^2}{1 - \bar{\zeta}z}\right)^k\end{aligned}$$

which coincides with $-g_{\mathbb{D}}(\zeta, z) + \log|\zeta|$ for $p = 0$. This kernel is suitable for measures $\nu \in \mathcal{M}^+(\mathbb{D})$ satisfying

$$\int_0^{1^-} (1-t)^{p+1} d\nu^{\text{rad}}(t) + \int_0^{1/2} \frac{\nu^{\text{rad}}(t)}{t} dt < +\infty. \quad (1.5)$$

- (H₂) The subharmonic *Horowitz's kernel* (see [18], [17]) on $\mathbb{D} \times \mathbb{D}$ is the function $h_2(\zeta, z) := \log|1 - (1 - B_\zeta(z))^2|$. This kernel is suitable for measures $\nu \in \mathcal{M}^+(\mathbb{D})$ satisfying $\int_0^{1^-} (1-t)^2 d\nu^{\text{rad}}(t) < +\infty$.
- (B_s) For $0 < s \leq 6$, the subharmonic *Beller's kernel* on $\mathbb{D} \times \mathbb{D}$ is the function $b_s(\zeta, z) := \log|1 - (1 - B_\zeta(z))^s|$ (see [3], [17]) which coincides with the Horowitz's kernel for $s = 2$ and with the Blaschke kernel for $s = 1$. This kernel is suitable for measures $\nu \in \mathcal{M}^+(\mathbb{D})$ satisfying $\int_0^{1^-} (1-t)^s d\nu^{\text{rad}}(t) < +\infty$.
- (\bar{B}_s) For $s \geq 1$, the subharmonic *Bomash's kernel* supported by $\mathbb{D} \setminus \{0\}$ is the function (see [19], [20])

$$\bar{b}_s(\zeta, z) := \log|1 - (1 - \bar{B}_\zeta(z))^s|.$$

This kernel is suitable for measures $\nu \in \mathcal{M}^+(\mathbb{D})$ satisfying (1.5) for $p = s - 1$. Below we will use a special case of subharmonic Bomash's kernel with $s = 2$:

$$\bar{b}_2(\zeta, z) := \log|1 - (1 - \bar{B}_\zeta(z))^2| = \log\left|1 - \left(\frac{1 - |\zeta|^2}{1 - \bar{\zeta}z}\right)^2\right| \quad (1.6a)$$

$$= \log(|\bar{B}_\zeta(z)||2 - \bar{B}_\zeta(z)|) = \log\frac{|\zeta||\zeta - z||2 - |\zeta|^2 - \bar{\zeta}z|}{|1 - \bar{\zeta}z|^2}. \quad (1.6b)$$

This kernel is suitable for measures $\nu \in \mathcal{M}^+(\mathbb{D})$ satisfying

$$\int_0^{1^-} (1-t)^2 d\nu^{\text{rad}}(t) + \int_0^{1/2} \frac{\nu^{\text{rad}}(t)}{t} dt < +\infty. \quad (1.7)$$

²For $q > p$, by definition, $\sum_{k=q}^p \cdots := 0$, $\prod_{k=q}^p \cdots := 1$. Similarly, $\sum_{k \in \emptyset} \cdots := 0$, $\prod_{k \in \emptyset} \cdots := 1$.

(K₁) The subharmonic *Korenblum's kernel* on $(\mathbb{D} \setminus \{0\}) \times \mathbb{D}$ (see [2], [17]) is the function $k_1(\zeta, z) := \log|B_\zeta(z)| + \log \frac{1}{|\zeta|} \operatorname{Re} \frac{\zeta/|\zeta| + z}{\zeta/|\zeta| - z}$. This kernel is suitable for measures $\nu \in \mathcal{M}^+(\mathbb{D})$ satisfying (1.7).

Note that for each measure $\nu \in \mathcal{M}(\mathbb{D})$ we can select subharmonic kernel (see Dzhrbashyan's factorization theory in [14], [15], [17]) which is suitable for ν and optimal in a certain sense.

3. Let $\Omega = \mathbb{C}$, $r_0 > 0$. The following functions are subharmonic kernels:

(E_q) For an integer $q \geq 0$, the subharmonic *Hadamard-Weierstrass kernel of genus q* on $(\mathbb{C} \setminus \{0\}) \times \mathbb{C}$ is the function

$$e_q(\zeta, z) := \log \left| 1 - \frac{z}{\zeta} \right| + \sum_{k=0}^q \frac{1}{k} \operatorname{Re} \frac{z^k}{\zeta^k}.$$

This kernel is suitable for measures $\nu \in \mathcal{M}^+(\mathbb{C})$ satisfying

$$\int_0^1 \frac{\nu^{\text{rad}}(t)}{t^{q+1}} dt + \int_1^{+\infty} \frac{\nu^{\text{rad}}(t)}{t^{q+2}} dt < +\infty.$$

(W) Let $\{q_n\}$ be a sequence of nonnegative integer, and let $\{r_n\}$ be an increasing sequence of positive numbers, $n \in \mathbb{N}$, $r_1 > r_0$. The subharmonic *Weierstrass kernel* (with respect to these two sequences) on $\mathbb{C} \times \mathbb{C}$ is the function $w(\zeta, z) := e_{q_n}(\zeta, z)$ when $r_{n-1} \leq |\zeta| < r_n$, $n \in \mathbb{N}$, and $w(\zeta, z) := \log|\zeta - z|$ when $|\zeta| < r_0$. This kernel is suitable for measures $\nu \in \mathcal{M}^+(\mathbb{C})$ satisfying (see [1, Theorem 4.1])

$$\sum_{n=1}^{\infty} \int_{r_{n-1}}^{r_n} \frac{\nu^{\text{rad}}(t)}{t^{q_n+2}} dt < +\infty.$$

Note also that, for each measure $\nu \in \mathcal{M}^+(\mathbb{C})$, we can select subharmonic kernel (see Dzhrbashyan's factorization theory in [21] and the survey [22]) which is suitable for ν and optimal in a certain sense.

The following Proposition 1.1 is a global version of Riesz Decomposition Theorem for subharmonic functions [1], [12].

Proposition 1.1. *Let k be a subharmonic kernel on $B \times \Omega$. Suppose a measure $\nu \in \mathcal{M}^+(\Omega)$ is concentrated in B . If the kernel k is suitable for ν , then the function*

$$U_k^\nu(z) := \int_{\Omega} k(\zeta, z) d\nu(\zeta)$$

is subharmonic on Ω with the Riesz measure ν . In particular, every function $M \in SH(\Omega)$ with the Riesz measure $\nu_M = \nu$ can be decomposed as

$$M = U_k^\nu + H, \quad \text{where } H \in \text{Harm}(\Omega). \quad (1.8)$$

The proof is omitted here. It can be obtained from [23, Theorem 2.6.5].

1.2 Main results

We write $L_{\text{loc}}^1(\Omega)$ for the set of all functions $F: \Omega \rightarrow [-\infty, +\infty]$ that are locally integrable with respect to m . A sequence $\{w_n\}$ from $L_{\text{loc}}^1(\Omega)$ is convergent in $L_{\text{loc}}^1(\Omega)$ if there exists a function $w \in L_{\text{loc}}^1(\Omega)$ such that $\lim_{n \rightarrow \infty} \int_K |w_n - w| dm \rightarrow 0$ for any compact subset $K \Subset \Omega$.

We denote by $\text{dist}(z, \partial\Omega)$ the Euclidean distance from $z \in \Omega$ to $\partial\Omega$.

In what follows the measure $m^{(r)} \in \mathcal{M}^+(\mathbb{C})$ is obtained from Lebesgue measure m by restricting it to the disk $D(r)$ and normalizing so that $m^{(r)}(D(r)) = 1$, i. e.,

$$m^{(r)} := \frac{1}{m^{\text{rad}}(r)} m \Big|_{D(r)}. \quad (1.9)$$

We have

$$v(z) \leq (v * m^{(r)})(z), \quad v \in SH(\Omega), \quad (1.10)$$

if $D(r) \Subset \Omega$ (here and below $*$ stands for the convolution operation).

Let $\sigma: \Omega \rightarrow (0, +\infty)$ be a function satisfying

$$0 < \sigma(z) < \text{dist}(z, \partial\Omega), \quad \forall z \in \Omega. \quad (1.11)$$

For $F \in L_{\text{loc}}^1(D)$, we put

$$F^{(\sigma)}(z) := \int_{D(\sigma(z))} F(z+w) dm^{(\sigma(z))}(w), \quad z \in \Omega. \quad (1.12)$$

We say that a function $F: \Omega \rightarrow [-\infty, +\infty]$ admits a harmonic minorant (a subharmonic minorant resp.) on Ω if and only if there is a $h \in \text{Harm}(\Omega)$ ($h \in SH(\Omega)$, $h \not\equiv -\infty$ on Ω , resp.) such that $h(z) \leq F(z)$ for all $z \in \Omega$.

For $a \in [-\infty, +\infty]$ (for a function $f: X \rightarrow [-\infty, +\infty]$ resp.), as usual, we put $a^+ := \max\{a, 0\}$ ($f^+(x) := \max\{f(x), 0\}$ resp.).

The following theorem is the main general result of our article.

Main Theorem. *Let $M \not\equiv -\infty$ be a subharmonic function with the Riesz measure ν_M on a domain $\Omega \subset \mathbb{C}$, $0 \in \Omega$. Suppose also that*

- i) *the function M is bounded on a subdomain $\Omega_1 \Subset \Omega$, and $0 \in \Omega_1$;*
- ii) *the measure ν_M is concentrated in a Borel subset $B \subset \Omega$, and subharmonic kernel k on $B \times \Omega$ is suitable for ν_M ;*

iii) for the function

$$Q_k^{\nu_M}(z) := \int_B (k(\zeta, 0) - k(\zeta, z))^+ d\nu_M(\zeta), \quad z \in \Omega, \quad (1.13)$$

there exists a majorizing function $Q \in L_{\text{loc}}^1(\Omega)$ on Ω , i. e., $Q_k^{\nu_M}(z) \leq Q(z)$ at almost all (with respect to Lebesgue measure m) points $z \in \Omega$.

Let u be a subharmonic function with the Riesz measure ν_u on Ω , and $u(0) \neq -\infty$. Then the following two assertions hold:

(Z) Let Ω_0 be a relative compact subdomain of Ω_1 containing the origin. If there exists a constant C such that the inequality

$$\int g_D(\zeta, 0) d\nu_u(\zeta) \leq \int g_D(\zeta, 0) d\nu_M(\zeta) + C \quad (1.14)$$

holds for each domain $D \in \mathcal{U}_0^d(\Omega; \Omega_0)$, then, for any $\sigma \in C(\Omega)$ satisfying (1.11), there exists a **harmonic function** h on Ω such that

$$u(z) + h(z) \leq M^{(\sigma)}(z) + Q^{(\sigma)}(z), \quad \forall z \in \Omega. \quad (1.15)$$

(S) If the difference $M - u$ admits³ a **subharmonic minorant** on Ω , then, for any $\sigma \in C(\Omega)$ satisfying (1.11), the function

$$M^{(\sigma)} + Q^{(\sigma)} - u \quad (1.16)$$

admits a **harmonic minorant** on Ω .

An important consequence from the Main Theorem is

Corollary 1.1. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain containing the origin. Let $M \not\equiv -\infty$ be a subharmonic continuous function on Ω with the Riesz measure ν_M . Suppose that the conditions i)–iii) of the Main Theorem are fulfilled. Then the following three assertions hold:*

(Z_{sc}) *Let $\Lambda = \{\lambda_n\}$ be a sequence in Ω , $0 \notin \Lambda$, and let Ω_0 be a relative compact subdomain of Ω containing the origin. If there exists a constant C such that the inequality*

$$\sum_n g_D(\lambda_n, 0) \leq \int g_D(\zeta, 0) d\nu_M(\zeta) + C \quad (1.17)$$

*holds for each domain $D \in \mathcal{U}_0^d(\Omega; \Omega_0)$, then, for any $\sigma \in C(\Omega)$ satisfying (1.11), the sequence Λ is a **zero set** for $\text{Hol}(\Omega; M^{(\sigma)} + Q^{(\sigma)})$.*

³If $M(z) = u(z) = -\infty$, then we put $(M - u)(z) = +\infty$.

(S_{sc}) If a sequence Λ in Ω is **zero subset** for the space $\text{Hol}(\Omega; M)$, then, for any $\sigma \in C(\Omega)$ satisfying (1.11), the sequence Λ is a **zero set** for the space $\text{Hol}(\Omega; M^{(\sigma)} + Q^{(\sigma)})$.

(M_{sc}) Suppose that $f = g/q$ is a meromorphic function on Ω where g and q belong to $\text{Hol}(\Omega; M)$; then, for any $\sigma \in C(\Omega)$ satisfying (1.11), there are holomorphic functions $g_0, q_0 \in \text{Hol}(\Omega; M^{(\sigma)} + Q^{(\sigma)})$ **without common zeros** such that $f = g_0/q_0$ on Ω .

Given $z \in \mathbb{D}$ and $\alpha > 0$, the set

$$\square_\alpha(z) := \{\zeta \in \mathbb{D} : |z| - \alpha(1 - |z|) \leq |\zeta|, |\arg \zeta - \arg z| \leq \alpha(1 - |z|)\}. \quad (1.18)$$

is called the *Carleson box* of relative size α with center at z .

Given $0 < \varepsilon < 1$ and $M : \mathbb{D} \rightarrow [-\infty, +\infty)$, we put

$$A_{M,\varepsilon}(z) := \frac{1}{2\pi} \int_0^{2\pi} M(z + \varepsilon(1 - |z|)e^{i\theta}) d\theta \quad (1.19)$$

if this integral there exists.

The following Theorem 1.1 is a special version of previous Corollary 1.1 in case that we use the subharmonic Bomash's kernel (1.6) from the item (\overline{B}_s) of Examples in the role of subharmonic kernel k on $(\mathbb{D} \setminus \{0\}) \times \mathbb{D}$.

Theorem 1.1. *Let $M \not\equiv -\infty$ be a subharmonic function M on \mathbb{D} with the Riesz measure ν_M . Suppose that the function M is bounded below on every $D \Subset \mathbb{D}$. Suppose also that the condition*

$$\int_0^{1^-} (1 - t)^2 d\nu_M^{\text{rad}}(t) < \infty \quad (1.20)$$

holds. By definition, put (see (1.18))

$$b_M^{[\alpha]}(z) := \frac{1}{(1 - |z|)^2} \int_{\square_\alpha(z)} (1 - |\zeta|)^2 d\nu_M(\zeta). \quad (1.21)$$

Under these conditions and notations, the following three assertions hold:

(Z _{\mathbb{D}}) Let $\Lambda = \{\lambda_n\}$ be a sequence in \mathbb{D} , and $a < 1$. If there exists a constant C such that the inequality (1.17) holds for each domain $D \Subset \mathcal{U}_0^d(\mathbb{D}; D(a))$, then, for any $0 < \varepsilon < 1$, the sequence Λ is a **zero set** for the space

$$\text{Hol}(\mathbb{D}; 2A_{M,\varepsilon} - M + C_\varepsilon \cdot b_M^{[\alpha]}), \quad (1.22)$$

where C_ε is a positive constant dependent only on ε , and α is an absolute constant.

- (S $_{\mathbb{D}}$) If a sequence Λ in \mathbb{D} is **zero subset** for the space $\text{Hol}(\mathbb{D}; M)$, then, for any $0 < \varepsilon < 1$, the sequence Λ is a **zero set** for the space (1.22).
- (M $_{\mathbb{D}}$) If $f = g/q$ is a meromorphic function on \mathbb{D} , and $g, q \in \text{Hol}(\mathbb{D}; M)$, then, for any $0 < \varepsilon < 1$, there are holomorphic functions g_0, q_0 **without common zeros** on \mathbb{D} such that $f = g_0/q_0$ on \mathbb{D} , and the functions g_0, q_0 belong to the space (1.22).

2 Bases of general approach

2.1 The balayage:

Arens–Singer and Jensen measures and functions

Let H be a convex subcone in $SH(\Omega)$, $z \in \Omega$. Let $\delta, \mu \in \mathcal{M}^+(\Omega)$ are measures with compact support on Ω . We write $\delta \prec_H \mu$ and say that μ is a *balayage of δ with respect to H* if

$$\int h d\delta \leq \int h d\mu, \quad \forall h \in H. \quad (2.1)$$

Given $z \in \mathbb{C}$, we denote by δ_z the Dirac measure at the point $z \in \mathbb{C}$, i. e. $\text{supp } \delta_z = \{z\}$, $\delta_z(\mathbb{C}) = 1$.

Definition 2 ([24]–[33]). A measure μ with compact support in $\Omega \ni z$ is called the *Jensen (Arens–Singer, or representing resp.) measure* for z on Ω if $\delta_z \prec_H \mu$ where $H = SH(\Omega)$ ($H = \text{Harm}(\Omega)$ resp.), i. e., if and only if

$$h(z) \leq \int h d\mu, \quad \forall h \in SH(\Omega) \quad \left(h(z) = \int h d\mu, \quad \forall h \in \text{Harm}(\Omega) \text{ resp.} \right). \quad (2.2)$$

Denote by $\mathcal{J}_z(\Omega)$ ($\mathcal{AS}_z(\Omega)$ resp.) the class of all Jensen (Arens–Singer resp.) measures for z . If $z = 0 \in \Omega$, then $\mathcal{J}(\Omega) := \mathcal{J}_0(\Omega)$, $\mathcal{AS}(\Omega) := \mathcal{AS}_0(\Omega)$.

Evidently, $\mathcal{J}_z(\Omega) \subset \mathcal{AS}_z(\Omega)$, and every $\mu \in \mathcal{AS}_z(\Omega)$ is a probability measure, i. e.,

$$\mu(\Omega) = 1. \quad (2.3)$$

A subclass of $\mathcal{J}_z(\Omega)$ is the class $\mathcal{H}_z(\Omega)$ ($\mathcal{H}_z^{\text{reg}}(\Omega)$ resp.) of all harmonic measures $\omega_D(z, \cdot)$ for domains (for regular for the Dirichlet problem domains resp.) $D \Subset \Omega$ at the point z . We set $\mathcal{H}(\Omega) := \mathcal{H}_0(\Omega)$, $\mathcal{H}^{\text{reg}}(\Omega) := \mathcal{H}_0^{\text{reg}}(\Omega)$. The *potential of a measure $\mu \in \mathcal{AS}(\Omega)$* is defined to be function

$$V_\mu(\zeta) := \int \log |z - \zeta| d\mu(z) - \log |\zeta| = \int \log \left| 1 - \frac{z}{\zeta} \right| d\mu(z), \quad \zeta \in \mathbb{C} \setminus \{0\}. \quad (2.4)$$

Let $\mu \in \mathcal{AS}(\Omega)$. For any function $u_\nu \in SH(\Omega)$ with the Riesz measure ν_u the generalized Poisson–Jensen formula [27, Proposition 1.2] holds:

$$u_\nu(0) = \int_{\Omega} u_\nu \, d\mu - \int_{\Omega} V_\mu \, d\nu. \quad (2.5)$$

Proposition 2.1. *Let V_μ be the potential (2.4) of a measure $\mu \in \mathcal{AS}(\Omega)$, and let k be a subharmonic kernel on $B \times \Omega$. Then, for every $\zeta \in B \setminus \{0\}$, the potential V_μ can be represented in the form*

$$V_\mu(\zeta) = \int_{\Omega} k(\zeta, z) \, d\mu(z) - k(\zeta, 0) = \int_{\Omega} (k(\zeta, z) - k(\zeta, 0)) \, d\mu(z). \quad (2.6)$$

Proof. By (1.2), for every $\zeta \in B \setminus \{0\}$, we have representations

$$k(\zeta, z) = \log |\zeta - z| + h(\zeta, z), \quad k(\zeta, 0) = \log |\zeta| + h(\zeta, 0), \quad (2.7)$$

where $h(\zeta, z)$ is harmonic function of $z \in \Omega$. Hence, for $\mu \in \mathcal{AS}(\Omega)$, we obtain

$$\begin{aligned} & \int_{\Omega} (k(\zeta, z) - k(\zeta, 0)) \, d\mu(z) \\ & \stackrel{(2.7)}{=} \int_{\Omega} (\log |\zeta - z| + h(\zeta, z) - \log |\zeta| - h(\zeta, 0)) \, d\mu(z) \\ & \stackrel{(2.3)}{=} \int_{\Omega} (\log |\zeta - z| - \log |\zeta|) \, d\mu(z) + \int_{\Omega} h(\zeta, z) \, d\mu(z) - h(\zeta, 0) \\ & \stackrel{(2.4), (2.2)}{=} V_\mu(\zeta) + h(\zeta, 0) - h(\zeta, 0) = V_\mu(\zeta) \end{aligned}$$

for each $\zeta \in B \setminus \{0\}$, as desired. •

Definition 3 ([24]–[29]). A function $V \in SH(\Omega \setminus \{0\})$ will be called a *Arens–Singer* (or *representing*) *function* on Ω if this function satisfies the following two conditions:

(1) There is a set $K \Subset \Omega$ such that $V(\zeta) \equiv 0$ for all $\zeta \in \Omega \setminus K$;

(2) $\limsup_{\zeta \rightarrow 0} \frac{V(\zeta)}{-\log |\zeta|} \leq 1$.

If V satisfies also the condition

(3) $V(\zeta) \geq 0$ for all $\zeta \in \Omega \setminus \{0\}$,

then we call V a *Jensen function* on Ω . Denote by $\mathcal{P}_{AS}(\Omega)$ and $\mathcal{P}_J(\Omega)$ the classes of all Arens–Singer and Jensen functions respectively.

Evidently, $\mathcal{P}_J(\Omega) \subset \mathcal{P}_{AS}(\Omega)$. The class $\mathcal{P}_{\mathcal{H}}(\Omega)$ ($\mathcal{P}_{\mathcal{H}}^{\text{reg}}(\Omega)$ resp.) of all extended Green functions $g_D(\cdot, 0)$ for domains $D \Subset \Omega$ (for regular for the Dirichlet problem domains $D \Subset \Omega$ resp.) is a subclass of $\mathcal{P}_J(\Omega)$.

Proposition 2.2 ([27, Duality Theorem]). *The map $\mathcal{P}: \mu \rightarrow V_\mu$ is an affine⁴ bijection from $\mathcal{AS}(\Omega)$ onto $\mathcal{P}_{AS}(\Omega)$, and from $\mathcal{J}(\Omega)$ onto $\mathcal{P}_J(\Omega)$.*

Note that, according to Proposition 2.2, for $V \in \mathcal{P}_{AS}(\Omega)$ the condition (2) is equivalent to the condition (see [12, Theorem 3.1.2])

$$(2') \quad V(\zeta) \leq -\log |\zeta| + O(1) \text{ as } \zeta \rightarrow 0.$$

Let $\Omega_0 \Subset \Omega$ be a domain containing the origin.

Denote by $\mathcal{AS}^{\Omega_0}(\Omega)$ ($\mathcal{J}^{\Omega_0}(\Omega)$ resp.) the set of all Arens–Singer (Jensen resp.) measures μ on Ω such that $\Omega_0 \cap \text{supp } \mu = \emptyset$.

Denote by $\mathcal{P}_{AS}^{\Omega_0}(\Omega)$ ($\mathcal{P}_J^{\Omega_0}(\Omega)$ resp.) the set of all Arens–Singer (Jensen resp.) functions V that is harmonic on $\Omega_0 \setminus \{0\}$ such that (cf. (2'))

$$V(\zeta) = -\log |\zeta| + O(1), \quad \zeta \rightarrow 0. \quad (2.8)$$

Proposition 2.3. *The map $\mathcal{P}: \mu \rightarrow V_\mu$ of Proposition 2.2 is an affine bijection from $\mathcal{AS}^{\Omega_0}(\Omega)$ onto $\mathcal{P}_{AS}^{\Omega_0}(\Omega)$, and from $\mathcal{J}^{\Omega_0}(\Omega)$ onto $\mathcal{P}_J^{\Omega_0}(\Omega)$.*

The proof of Proposition 2.3 follows from Proposition 2.2 and [27, Proposition 1.4, (4)].

We recall the notion of classical balayage [34]. Given a bounded regular for the Dirichlet problem domain $\Omega_1 \Subset \Omega$ and a measure $\mu \in \mathcal{M}^+(\Omega)$, the classical balayage of μ relative to Ω_1 is the measure $\mu^{\Omega_1} \in \mathcal{M}^+(\Omega)$ defined by

$$\mu^{\Omega_1}(E) := \mu(E \setminus \Omega_1) + \int_{\Omega_1} \omega_{\Omega_1}(z, E) d\mu(z), \quad (\text{Borel } E \Subset \Omega). \quad (2.9)$$

In particular, $\Omega_1 \cap \text{supp } \mu^{\Omega_1} = \emptyset$, and $\mu \prec_{SH(\Omega)} \mu^{\Omega_1}$ [32, Lemma 6.4]. Hence

$$\mathcal{AS}^{\Omega_1}(\Omega) = \{\mu^{\Omega_1} : \mu \in \mathcal{AS}(\Omega)\}, \quad \mathcal{J}^{\Omega_1}(\Omega) = \{\mu^{\Omega_1} : \mu \in \mathcal{J}(\Omega)\} \quad (2.10)$$

for regular subdomains $\Omega_1 \Subset \Omega$ containing the origin.

Let $D \Subset \mathbb{C}$ be a regular domain containing the origin. For short, denote by ω_D the harmonic measure for D at the origin.

As before, δ_0 is the Dirac measure at 0.

Proposition 2.4 ([34, Ch. IV, § 1, and Theorem 4.12]). *Let $D \Subset \mathbb{C}$, $\Omega' \Subset \mathbb{C}$, and $D \cup \Omega'$ are regular domains containing the origin. Then*

$$\delta_0^{\Omega' \cup D} = \omega_{\Omega' \cup D} = (\omega_D)^{\Omega' \cup D} = \omega_{\Omega'} - (\omega_{\Omega'} \mid_D) + (\omega_{\Omega'} \mid_D)^{\Omega' \cup D}. \quad (2.11)$$

⁴It means that $\mathcal{P}(\alpha\mu_1 + (1-\alpha)\mu_2) = \alpha\mathcal{P}(\mu_1) + (1-\alpha)\mathcal{P}(\mu_2)$ for $0 \leq \alpha \leq 1$.

2.2 On the existence of (sub)harmonic minorants

Proposition 2.5 (cf. [26, Proposition 7.1]). *Let $F: \Omega \rightarrow [-\infty, +\infty]$ be a Borel-measurable function. If F admits a harmonic minorant, then*

$$-\infty < \inf \left\{ \int_{\Omega} F \, d\mu : \mu \in \mathcal{AS}(\Omega) \right\}. \quad (2.12)$$

If F admits a subharmonic minorant, and F is bounded below in a certain neighbourhood of origin, then

$$-\infty < \inf \left\{ \int_{\Omega} F \, d\mu : \mu \in \mathcal{J}(\Omega) \right\}. \quad (2.13)$$

Evidently, the class $\mathcal{AS}(\Omega)$ in (2.12) can be replaced by $\mathcal{J}(\Omega)$, $\mathcal{H}(\Omega)$, $\mathcal{H}^{\text{reg}}(\Omega)$, and we can substitute $\mathcal{H}(\Omega)$ or $\mathcal{H}^{\text{reg}}(\Omega)$ for $\mathcal{J}(\Omega)$ in (2.13).

Proof. Let h be a Borel-measurable function on Ω and $F \geq h$ on Ω . Then $\int F \, d\mu \geq \int h \, d\mu$ for every $\mu \in \mathcal{M}^+(\Omega)$. If the minorant h is harmonic, then $\int h \, d\mu \geq h(0)$ for all $\mu \in \mathcal{AS}(\Omega)$ according to (2.2), and (2.12) is proved.

Let us assume that F is bounded below on $D(2\varepsilon) \Subset \Omega$, and the minorant h is subharmonic. There exists a subharmonic function h_ε which coincides with h on $\Omega \setminus D(\varepsilon)$ and harmonic on $D(\varepsilon)$ [1, Theorem 2.18]. Hence the difference $h_\varepsilon - C$ is a subharmonic minorant for F if a constant C is sufficiently large. Therefore, $\int F \, d\mu \geq \int (h_\varepsilon - C) \, d\mu \geq h_\varepsilon(0) - C$ for all $\mu \in \mathcal{J}(\Omega)$ according to Definition 2, and (2.13) is proved. •

We need another treatments of Proposition 2.5 for the cases special functions F representing by the difference

$$F = M - u, \quad u \in SH(\Omega), \quad u(0) \neq -\infty. \quad (2.14)$$

Here the functions M and u can take on the value $-\infty$ at a point $z \in \Omega$ simultaneously if $M \in SH(\Omega)$ or $M \in L^1_{\text{loc}}(\Omega)$. In this case we will be consider that $F(z) = +\infty$. The following Proposition 2.6 is an easy corollary of Propositions 2.5, 2.2, and of generalized Poisson–Jensen formula (2.5).

Proposition 2.6. *Let $M: \Omega \rightarrow [-\infty, +\infty]$ be a Borel-measurable function, and u be a subharmonic function on Ω with the Riesz measure ν_u .*

If the function $F := M - u$ from (2.14) admits a harmonic minorant on Ω , then there is a constant C such that

$$\int_{\Omega} V_{\mu} \, d\nu_u \leq \int_{\Omega} M \, d\mu + C \quad (2.15)$$

for all $\mu \in \mathcal{AS}(\Omega)$ where V_μ is the potential of μ (see definition (2.4)). Besides, if $M \in SH(\Omega)$, then there is a constant C such that

$$\int_{\Omega} V \, d\nu_u \leq \int_{\Omega} V \, d\nu_M + C \quad (2.16)$$

for all Arens–Singer functions V on Ω .

If $F = M - u$ from (2.14) admits a subharmonic minorant, and M is bounded below in a certain neighbourhood of origin, then there is a constant C such that (2.15) holds for all $\mu \in \mathcal{J}(\Omega)$. Besides, if $M \in SH(\Omega)$, then there is a constant C such that (2.16) holds for all $V \in \mathcal{P}_{\mathcal{J}}(\Omega)$.

Let $\{u_n\}$ be a sequence of functions subharmonic in Ω and uniformly locally bounded above (that is, uniformly bounded above on every $G \Subset \Omega$) in Ω . We denote by $\limsup^* u_n$ the upper semicontinuous regularization of the (pointwise) upper limit of $\{u_n\}$, which is, in turn, a subharmonic function.

The following Theorem is in a sense converse to Proposition 2.5.

Theorem 2.1 ([26, Theorem 7.1]). *Let $H \subset SH(\Omega)$ be a convex cone. Assume that the cone H contains a negative (≤ 0) function, and let $F: \Omega \rightarrow [-\infty, +\infty]$ be a function that belongs to $L^1_{\text{loc}}(\Omega)$.*

Let us suppose that for any $K \Subset \Omega$ and any constant C there is a $h \in H$ such that $h \leq C$ on K , and at least one of the following conditions holds:

(L*) *if a sequence $\{h_n\}$, $h_n \in H$, is locally bounded above, then the function $\limsup^* h_n$ belongs to H ,*

(CL) *the cone H is sequentially closed in $L^1_{\text{loc}}(\Omega)$.*

If

$$-\infty < \inf \left\{ \int_{\Omega} F \, d\mu: \delta_0 \prec_H \mu, \mu \in \mathcal{M}_{\text{ac}}^+(\Omega) \right\}, \quad (2.17)$$

where δ_0 is the Dirac measure at 0, then for any function $\sigma \in C(\Omega)$ satisfying (1.11), there is a function $h \in H$ such that⁵ $h \not\equiv -\infty$ on Ω , and

$$h(z) \leq F^{(\sigma)}(z) := \int_{D(\sigma(z))} F(z+w) \, dm^{(\sigma(z))}(w), \quad \forall z \in \Omega. \quad (2.18)$$

The space $H = \text{Harm}(\Omega)$ and the cone $H = SH(\Omega)$ satisfy the conditions of Theorem 2.1 (see [25]–[26] for example).

Results more general than Theorem 2.1 were proved for abstract cones H of vector lattices [25, Theorems 5.1, 6.1], and, for $H = SH(\Omega)$, in [31,

⁵In [25], we proposed by definition that the cone $SH(\Omega)$ is not containing the function identically equal to $-\infty$.

Corollary 1.7]. They require a certain preparation, and we do not use their here.

We also need another treatments of theorem 2.1 for the cases $H = \text{Harm}(\Omega)$ and $H = SH(\Omega)$, and for special functions F representing by the difference (2.14) where $M \in L^1_{\text{loc}}(\Omega)$ or $M \in SH(\Omega)$.

The following Theorem 2.2 is an earlier version of Theorem 2.1 which joins Proposition 2.5 with Theorem 2.1 for special functions (2.14) with $M \in C(\Omega)$.

Theorem 2.2 ([24, Main Theorem]). *Let $u \in SH(\Omega)$ and $M \in C(\Omega)$. The difference (2.14) admits a harmonic (subharmonic resp.) minorant on Ω if and only if there is a constant C such that $\int_{\Omega} u \, d\mu \leq \int_{\Omega} M \, d\mu + C$ for all $\mu \in \mathcal{AS}(\Omega)$ (for all $\mu \in \mathcal{J}(\Omega)$ resp.).*

We also need an other version of Theorem 2.2 in the case that the function M in (2.14) belongs to $L^1_{\text{loc}}(\Omega)$ or $SH(\Omega)$.

Theorem 2.3. *Let $u \not\equiv -\infty$ be a subharmonic function with the Riesz measure ν_u , and $u(0) \neq -\infty$. Let $M \in L^1_{\text{loc}}(\Omega)$ be a function that is bounded on a subdomain $\Omega_1 \Subset \Omega$ containing the origin.*

If there are a subdomain $\Omega_0 \Subset \Omega_1$ and constant C such that the inequality

$$\int_{\Omega} u \, d\mu \leq \int_{\Omega} M \, d\mu + C \quad \text{or} \quad \int_{\Omega} V_{\mu} \, d\nu_u \leq \int_{\Omega} M \, d\mu + C \quad (2.19)$$

holds for every $\mu \in \mathcal{AS}^{\Omega_0}(\Omega) \cap \mathcal{M}_{\text{ac}}(\Omega)$ (for every $\mu \in \mathcal{J}^{\Omega_0}(\Omega) \cap \mathcal{M}_{\text{ac}}(\Omega)$ resp.) where V_{μ} is the potential (2.4) of μ , then, for any function $\sigma \in C(\Omega)$ satisfying (1.11), there exists a harmonic (subharmonic resp.) function h ($h \not\equiv -\infty$ resp.) on Ω such that (see (1.12))

$$u(z) + h(z) \leq M^{(\sigma)}(z), \quad \forall z \in \Omega. \quad (2.20)$$

In addition, if $M \in SH(\Omega)$, then the inequality (2.19) can be replace by

$$\int_{\Omega} V \, d\nu_u \leq \int_{\Omega} V \, d\nu_M + C \quad (2.21)$$

for every function $V \in \mathcal{P}_{AS}^{\Omega_0}(\Omega) \cap C(\Omega)$ (function $V \in \mathcal{P}_J^{\Omega_0}(\Omega) \cap C(\Omega)$ resp.).

Proof. Here and later we need the following elementary

Lemma 2.1. *Let $\Omega_0 \Subset \Omega_1$ are subdomains of Ω , and $0 \in \Omega_0$. Then there is a domain $\Omega' \in \mathcal{U}_0^d(\Omega_1; \Omega_0)$ such that $\Omega_0 \Subset \Omega' \Subset \Omega_1$. In particular, the domain Ω' is regular for the Dirichlet problem.*

By Lemma 2.1 we can assume that Ω_1 is regular for the Dirichlet problem, $m(\partial\Omega_1) = 0$, and M is bounded in an open neighbourhood of the closure $\overline{\Omega}_1$. Then we can choose a sufficiently small constant $\varepsilon > 0$ and a constant $c \geq 0$ such that

$$|M(z+w)| \leq c, \quad \forall z \in \Omega_1, \quad \forall w \in D(\varepsilon), \quad (2.22M)$$

$$\bigcup_{z \in \Omega_1} D(z, \varepsilon) \Subset \Omega, \quad \Omega_0 \cap \left(\bigcup_{z \in \partial\Omega_1} D(z, \varepsilon) \right) = \emptyset. \quad (2.22\Omega)$$

We put $F = M - u$, and $H = SH(\Omega)$ or $H = \text{Harm}(\Omega)$. According to generalized Poisson–Jensen formula (2.5) and Proposition 2.3, it follows from (2.19) that the inequality

$$\int F \, d\mu > -C \quad (2.23)$$

holds for every measure $\mu \in \mathcal{M}_{\text{ac}}(\Omega)$ such that $\delta_0 \prec_H \mu$ and $\Omega_0 \cap \text{supp } \mu = \emptyset$.

Let μ be an arbitrary measure that belongs to $\mathcal{M}_{\text{ac}}^+(\Omega)$, and $\delta_0 \prec_H \mu$. Denote by μ_1 the restriction of μ to Ω_1 . We have the representation

$$\mu = \mu_1 + \mu_\infty, \quad \mu_1, \mu_\infty \in \mathcal{M}_{\text{ac}}^+(\Omega). \quad (2.24)$$

The measure $\mu' := \mu_1^{\Omega_1} * m^{(\varepsilon)} \in \mathcal{M}_{\text{ac}}^+(\Omega)$ is the balayage of μ_1 with respect to $SH(\Omega)$ (see (2.10) and [26, Lemma 7.1]). Hence $\delta_0 \prec_H (\mu' + \mu_\infty) \in \mathcal{M}_{\text{ac}}^+(\Omega)$,

$$\int u \, d\mu_1 \leq \int u \, d\mu', \quad (2.25)$$

and, by constructions (2.9) and (2.22 Ω), $\Omega_0 \cap \text{supp}(\mu' + \mu_\infty) = \emptyset$. The inequality (2.23) holds for all such measures. Thus,

$$\int F \, d(\mu' + \mu_\infty) \geq -C. \quad (2.26)$$

Hence we get

$$\begin{aligned} \int F \, d\mu &\stackrel{(2.24)}{=} \int F \, d(\mu' + \mu_\infty) + \int F \, d\mu_1 - \int (M - u) \, d\mu' \\ &\stackrel{(2.26)}{\geq} -C - \left(\int u \, d\mu_1 - \int u \, d\mu' \right) + \int_{\overline{\Omega}_1} M \, d\mu_1 - \int_{\overline{\Omega}_1} M \, d\mu' \\ &\stackrel{(2.25)}{\geq} -C - \int_{\overline{\Omega}_1} |M| \, d(\mu_1 + \mu') \stackrel{(2.22M), (2.3)}{\geq} -C - 2c. \end{aligned}$$

The last inequality implies the condition (2.17) of Theorem 2.1. It follows that there is a function $h \in H$ such that (2.18) holds. Under our notation, (2.18) coincides with (2.20).

For the case (2.21), by Proposition 2.3, we see that the map \mathcal{P} acts from $\mathcal{AS}^{\Omega_0}(\Omega) \cap \mathcal{M}_{\text{ac}}(\Omega)$ into $\mathcal{P}_{AS}^{\Omega_0}(\Omega) \cap C(\Omega)$ and from $\mathcal{AS}^{\Omega_0}(\Omega) \cap \mathcal{M}_{\text{ac}}(\Omega)$ into $\mathcal{P}_{AS}^{\Omega_0}(\Omega) \cap C(\Omega)$. Hence, (2.19) follows from (2.21) according to generalized Poisson–Jensen formula (2.5). \bullet

Remark. In Theorem 2.3 we may assume that the inequalities (2.19) are fulfilled only for all measures $\mu \in \mathcal{AS}^{\bar{\Omega}_0}(\Omega) \cap \mathcal{M}_{\text{ac}}(\Omega)$ (only for all measures $\mu \in \mathcal{J}^{\bar{\Omega}_0}(\Omega) \cap \mathcal{M}_{\text{ac}}(\Omega)$ resp.) where

$$\begin{aligned} \mathcal{AS}^{\bar{\Omega}_0}(\Omega) &:= \{\mu \in \mathcal{AS}(\Omega) : \bar{\Omega}_0 \cap \text{supp } \mu = \emptyset\}, \\ \mathcal{J}^{\bar{\Omega}_0}(\Omega) &:= \{\mu \in \mathcal{J}(\Omega) : \bar{\Omega}_0 \cap \text{supp } \mu = \emptyset\}. \end{aligned} \quad (2.27j)$$

It follows from an arbitrary rule for selection of $\Omega_0 \Subset \Omega_1$.

3 From Green functions to Jensen measures

The aim of this section is to prove that the conditions of statements (Z) and (S) of the Main Theorem give an united condition in terms of Jensen measures. For the statement (S) this condition is the following evident corollary of Proposition 2.6:

(J) *There exists a constant C such that*

$$\int_{\Omega} V_{\mu} d\nu_u \leq \int_{\Omega} M d\mu + C \quad (3.1)$$

for all measures $\mu \in \mathcal{J}^{\Omega_0}(\Omega)$.

In subsections 3.1 and 3.2 we show that the condition (J) also is fulfilled under the conditions of (Z).

3.1 A special approximation of Jensen measures

First we recall joint results of B. Cole and T. Ransford [32] about the approximation of Jensen measures by harmonic measures.

We write $\text{conv } A$ ($\overline{\text{conv}} A$ resp.) for the convex (closed convex resp.) hull of a set A .

Let S be an open subset or a compact subset of Ω . The space $C(S)$ is a Fréchet space with the topology of uniform convergence on compact subsets of S . The dual space $C(S)^*$ may be identified with the space of finite signed Borel measures on Ω of compact support. We use only weak*-topology in the space $C(S)^*$.

Theorem 3.1 ([31, Theorem 6.6]). $\mathcal{J}(\Omega) = \overline{\text{conv}}(\mathcal{H}(\Omega))$.

Proposition 3.1 ([31, Proposition 2.1], [27, Proposition 1.1]). *Let Ω_1 be a subdomain of Ω , $0 \in \Omega_1$. Then $\mathcal{J}(\Omega_1) \subset \mathcal{J}(\Omega)$. If $\mu \in \mathcal{J}(\Omega)$ and $\text{supp } \mu \subset \Omega_1$, and if each bounded component of $\mathbb{C} \setminus \Omega_1$ meets $\mathbb{C} \setminus \Omega$, then $\mu \in \mathcal{J}(\Omega_1)$.*

Proposition 3.2. *Let D be a subdomain of Ω , and $0 \in D \Subset \Omega$. Then there exists a increasing sequence of domains $D_n \in \mathcal{U}_0^d(D)$ such that the sequence of harmonic measures $\omega_{D_n}(0, \cdot)$ converges to $\omega_D(0, \cdot)$ in $C(\Omega)^*$.*

Proof. See [34, Theorems 4.15, 5.14]. •

Proposition 3.3. *Let Ω_0 and Ω_1 are subdomains of Ω such that*

$$0 \in \Omega_0 \Subset \Omega_1 \Subset \Omega. \quad (3.2)$$

Let K be a compact subset of Ω satisfying $K \cap \overline{\Omega_0} = \emptyset$. Then there are domains $\Omega' \in \mathcal{U}_0^d(\Omega_1; \Omega_0)$ and $\Omega'' \in \mathcal{U}_0^d(\Omega; \Omega_1)$ such that

$$\Omega_0 \Subset \Omega' \Subset \Omega_1 \Subset \Omega'', \quad K \subset \Omega_1 \setminus \overline{\Omega'}, \quad (3.3)$$

and each bounded component of $\mathbb{C} \setminus \Omega''$ meets $\mathbb{C} \setminus \Omega$.

The proof of Proposition 3.3 follows from Lemma 2.1.

Given domains $\Omega' \Subset \Omega_2$ containing 0, we write

$$\begin{aligned} \mathcal{H}^{\text{ud}}(\Omega_2) &:= \{\omega_D(0, \cdot) : D \in \mathcal{U}_0^d(\Omega_2)\}, \\ \mathcal{H}^{\text{ud}}(\Omega_2; \Omega') &:= \{\omega_D(0, \cdot) : D \in \mathcal{U}_0^d(\Omega_2; \Omega')\}. \end{aligned}$$

Proposition 3.4. *Let μ be a Jensen measure for 0 on Ω , and $K := \text{supp } \mu$. Under the notations and conditions of Proposition 3.3,*

$$\mu \in \overline{\text{conv}}\mathcal{H}^{\text{ud}}(\Omega''; \Omega') \quad (3.4)$$

where the closure is taken with respect to the weak-topology on $C(\Omega'')^*$.*

Proof. By Proposition 3.3 (for Ω'' instead of Ω) there exists one more domain $\Omega_2 \in \mathcal{U}_0^d(\Omega''; \Omega_1)$ such that $\Omega_1 \Subset \Omega_2 \Subset \Omega''$, each bounded component of $\mathbb{C} \setminus \Omega_2$ meets $\mathbb{C} \setminus \Omega''$, and $\text{supp } \mu \subset \Omega_2 \setminus \overline{\Omega'}$. By Proposition 3.1 (for Ω_2 instead of Ω_1) and Theorem 3.1 (for Ω_2 instead of Ω), the measure μ belongs to $\overline{\text{conv}}\mathcal{H}(\Omega_2)$ where the closure is taken with respect to weak*-topology of $C(\Omega_2)^*$. According to Proposition 3.2 we have

$$\overline{\text{conv}}\mathcal{H}(\Omega_2) = \overline{\text{conv}}\overline{\mathcal{H}^{\text{ud}}(\Omega_2)} = \overline{\text{conv}}\mathcal{H}^{\text{ud}}(\Omega_2)$$

in $C(\Omega_2)^*$. Therefore a net $\{\sigma_\gamma\} \subset \text{conv}\mathcal{H}^{\text{ud}}(\Omega_2)$, $\gamma \in \Gamma$, converges to μ in $C(\Omega_2)^*$. For each subscript γ there are a finitely many numbers $c_k(\gamma) > 0$ and domains $D_k(\gamma) \in \mathcal{U}_0^d(\Omega_2)$ such that

$$\sigma_\gamma = \sum_k c_k(\gamma) \omega_{D_k(\gamma)}, \quad \omega_{D_k(\gamma)} := \omega_{D_k(\gamma)}(0, \cdot), \quad \sum_k c_k(\gamma) = 1. \quad (3.5)$$

There is a domain Ω_- such that $\Omega' \Subset \Omega_- \Subset \Omega_1$ and $K \subset \Omega_2 \setminus \overline{\Omega_-}$. For every k, γ we can choose a domain $\Omega_k(\gamma) \in \mathcal{U}_0^d(\Omega_-; \Omega')$ such that the domain $D'_k(\gamma) := D_k(\gamma) \cup \Omega_k(\gamma) \in \mathcal{U}_0^d(\Omega_2; \Omega')$ is a regular domain, i. e., the complement $\mathbb{C} \setminus (D_k(\gamma) \cup \Omega_k(\gamma))$ has not of isolated points. Consider the restrictions

$$\omega'_{D'_k(\gamma)} := \omega_{D_k(\gamma)} \big|_{\Omega_k(\gamma)}, \quad \sigma'_\gamma := \sum_k c_k(\gamma) \omega'_{D'_k(\gamma)}. \quad (3.6)$$

Since $\text{supp } \mu \Subset \Omega_2 \setminus \overline{\Omega_-}$, the net $\{\sigma'_\gamma\}$ converges to the null measure in $C(\overline{\Omega'})^*$:

$$\sum_k c_k(\gamma) \omega'_{D'_k(\gamma)}(\overline{\Omega'}) \xrightarrow{\Gamma} 0. \quad (3.7)$$

Now we reconstruct the net $\{\sigma_\gamma\}$ with the help of balayage.

Denote by $(\omega'_{D'_k(\gamma)})^{D'_k(\gamma)}$ the classical balayage of $\omega'_{D'_k(\gamma)}$ relative to $D'_k(\gamma)$, and denote by $\omega''_{D'_k(\gamma)}$ the classical balayage of $\omega_{D_k(\gamma)}$ relative to $D'_k(\gamma)$. By Proposition 2.4 with $D = D_k(\gamma)$ each measure $\omega''_{D'_k(\gamma)}$ is exactly the harmonic measure $\omega_{D'_k(\gamma)}(0, \cdot)$. So, by definition (2.9) and equalities (2.11), we obtain

$$(\omega'_{D'_k(\gamma)})^{D'_k(\gamma)}(\Omega_2) = \omega'_{D'_k(\gamma)}(\Omega'), \quad (3.8)$$

$$\text{supp}(\omega'_{D'_k(\gamma)})^{D'_k(\gamma)} \subset \overline{\Omega_2}. \quad (3.8')$$

$$\omega_{D'_k(\gamma)}(0, \cdot) = \omega''_{D'_k(\gamma)} = (\omega'_{D'_k(\gamma)})^{D'_k(\gamma)} + (\omega_{D_k(\gamma)} - \omega'_{D'_k(\gamma)}). \quad (3.8'')$$

It follows from (3.7) and (3.8) that

$$\sum_k c_k(\gamma) (\omega'_{D'_k(\gamma)})^{D'_k(\gamma)}(\Omega_2) \xrightarrow{\Gamma} 0.$$

Hence, in view of (3.8'), (3.6), and (3.7), the nets

$$\sigma''_\gamma = \sum_k c_k(\gamma) (\omega'_{D'_k(\gamma)})^{D'_k(\gamma)}, \quad \sigma'_\gamma = \sum_k c_k(\gamma) \omega'_{D'_k(\gamma)}$$

converge to the null measure in $C(\overline{\Omega_2})^*$. It follows from the representations (3.8'') and (3.5) that the net

$$\sigma_\gamma^* := \sum_k c_k(\gamma) \omega_{D'_k(\gamma)}(0, \cdot) = \sigma''_\gamma + \sigma_\gamma - \sigma'_\gamma, \quad \Omega' \subset D'_k(\gamma) \Subset \Omega_2,$$

converges to μ in $C(\overline{\Omega_2})^*$, and all the more in $C(\Omega'')^*$. So, (3.4) is proved. •

3.2 The property (J) for the statement (Z)

Suppose that a Jensen measure μ belongs to $\mathcal{J}^{\bar{\Omega}_0}(\Omega)$ (see definition (2.27j)) for the same subdomain Ω_0 as in (Z). Put $K := \text{supp } \mu$. The function M of Main Theorem is bounded on the subdomain Ω_1 of Main Theorem satisfying (3.2). By Propositions 3.3 and 3.4, there are domains $\Omega', \Omega'' \in \mathcal{U}_0^d(\Omega_1; \Omega_0)$ such that (3.3) and (3.4) are fulfilled.

Put

$$d' := \text{dist}(\Omega_0, \partial\Omega'), \quad d'' := \text{dist}(\Omega'', \partial\Omega). \quad (3.9)$$

We can choose a number ε so that

$$0 < \varepsilon < \min\{d', d''\}. \quad (3.10)$$

Let $D \in \mathcal{U}_0^d(\Omega''; \Omega')$. According to (3.9) and (3.10), for any $w \in D(\varepsilon)$, the domain $D_w = \{z - w : z \in D\}$ belongs to the class $\mathcal{U}_0^d(\Omega; \Omega_0)$. Therefore, by the main condition of (Z), there is a constant C such that the inequality

$$\int_{\Omega} g_{D_w}(\zeta, 0) d\nu_u(\zeta) \leq \int_{\Omega} g_{D_w}(\zeta, 0) d\nu_M(\zeta) + C \quad (3.11)$$

holds for each domain $D \in \mathcal{U}_0^d(\Omega''; \Omega')$. Here the constant C is independent of D , w , and μ .

By the generalized Poisson–Jensen formula (2.5), the inequality (3.11) can be rewritten as

$$\int_{\Omega} u(z) d\omega_{D_w}(0, z) \leq \int_{\Omega} M(z) d\omega_{D_w}(0, z) + C' \quad (3.12)$$

where the constant $C' = C - u(0) - M(0)$ is independent of $D \in \mathcal{U}_0^d(\Omega''; \Omega')$, $w \in D(\varepsilon)$, μ , and ε under condition (3.10). If we replace $z - w$ by a new variable in (3.12), then

$$\int_{\Omega} u(z + w) d\omega_D(0, z) \leq \int_{\Omega} M(z + w) d\omega_D(0, z) + C' \quad (3.13)$$

for all $D \in \mathcal{U}_0^d(\Omega''; \Omega')$ and $w \in D(\varepsilon)$ where the constant C' is independent of μ and ε of (3.10). Integrating the last inequality with respect to the probability measure $m^{(\varepsilon)}$ (see (1.9)), by Fubini's theorem, we obtain

$$\begin{aligned} \int_{\Omega} (u * m^{(\varepsilon)})(z) d\omega_D(0, z) &= \int_{D(\varepsilon)} \int_{\Omega} u(z + w) d\omega_D(0, z) dm^{(\varepsilon)}(w) \\ &\stackrel{(3.13)}{\leq} \int_{D(\varepsilon)} \int_{\Omega} M(z + w) d\omega_D(0, z) dm^{(\varepsilon)}(w) + \int_{D(\varepsilon)} C' dm^{(\varepsilon)}(w) \\ &= \int_{\Omega} (M * m^{(\varepsilon)})(z) d\omega_D(0, z) + C'. \end{aligned}$$

Hence, under notation (1.12),

$$\int_{\Omega} u^{(\varepsilon)}(z) d\omega_D(0, z) \leq \int_{\Omega} M^{(\varepsilon)}(z) d\omega_D(0, z) + C'$$

for all domains $D \in \mathcal{U}_0^d(\Omega''; \Omega')$ where the functions $u^{(\varepsilon)}$ and $M^{(\varepsilon)}$ are well defined and continuous on Ω'' . According to (3.4) of Proposition 3.4, it implies that

$$\int_{\Omega} u^{(\varepsilon)}(z) d\mu(z) \leq \int_{\Omega} M^{(\varepsilon)}(z) d\mu(z) + C'.$$

In view of (1.10), we have

$$\int_{\Omega} u d\mu \leq \int_{\Omega} M^{(\varepsilon)} d\mu + C'.$$

If ε tend to 0, then the decreasing net $\{M^{(\varepsilon)}\}$ of continuous functions tend pointwise to M on Ω'' since M is subharmonic. Thus,

$$\int_{\Omega} u d\mu \leq \int_{\Omega} M d\mu + C'. \quad (3.14)$$

But the measure $\mu \in \mathcal{J}^{\bar{\Omega}_0}(\Omega)$ was any given, and the constant C' is independent of μ . Therefore, the inequality (3.14) is fulfilled for every measure $\mu \in \mathcal{J}^{\bar{\Omega}_0}(\Omega)$. By the generalized Poisson–Jensen formula (2.5), the inequality (3.14) implies the inequality

$$\int_{\Omega} V_{\mu} d\nu_u \leq \int_{\Omega} M d\mu + (C' - u(0)) \quad (3.15)$$

for every measure $\mu \in \mathcal{J}^{\bar{\Omega}_0}(\Omega)$ where V_{μ} is potential (2.4) of μ , and the constant $C'' := C' - u(0)$ is independent of μ . But the inequality (3.15) is exactly (3.1) with constant C'' instead of C . Besides, here we can replace the upper index of $\mathcal{J}^{\bar{\Omega}_0}(\Omega)$ by Ω_0 . The last follows from an arbitrary rule for selection of $\Omega_0 \Subset \Omega_1$. The property (J) is proved. •

Remark. For the deduction of (J) from (S) or (Z) we did not use a some representation of M by a subharmonic kernel. In other words, the conditions ii) and iii) of Main Theorem are unnecessary in order to prove (J).

4 The proofs of the Main Theorem

4.1 From Jensen measures and functions to the existence of harmonic minorant

The aim of this subsection is to prove that the condition (J) at the beginning of Section 3 implies the existence of harmonic minorant for the function (1.16). After Section 3 it gives both (S) and (Z).

Suppose that the condition (J) is fulfilled.

By the generalized Poisson–Jensen formula (2.5) and Proposition 2.3, it follows from the condition (J) that, for the constant $C' = C + M(0)$,

(J') *the inequality*

$$\int_{\Omega} V \, d\nu_u \leq \int_{\Omega} V \, d\nu_M + C' \quad (4.1)$$

holds for every Jensen function $V \in \mathcal{P}_J^{\Omega_0}(\Omega)$ (see above (2.8)).

Now let $\mu \in \mathcal{AS}^{\Omega_0}(\Omega) \cap \mathcal{M}_{ac}^+(\Omega)$ be a *Arens–Singer measure with the potential* V_{μ} (see (2.4)). By Proposition 2.3 the Arens–Singer function V_{μ} belongs to the class $\mathcal{P}_{AS}^{\Omega_0}(\Omega)$. Besides, by Definition 3, the function

$$V_{\mu}^+(\zeta) := \max\{V(\zeta), 0\}, \quad \zeta \in \mathbb{C} \setminus \{0\}, \quad (4.2)$$

is a *Jensen function* that belongs to the class $\mathcal{P}_J^{\Omega_0}(\Omega)$. Therefore the condition (J') implies that the inequalities

$$\int_{\Omega} V_{\mu} \, d\nu_u \leq \int_{\Omega} V_{\mu}^+ \, d\nu_u \leq \int_{\Omega} V_{\mu}^+ \, d\nu_M + C' \quad (4.3)$$

are fulfilled for all *Arens–Singer measures* $\mu \in \mathcal{AS}^{\Omega_0}(\Omega)$.

We shall need to estimate above the last integral

$$I_{\mu} := \int_{\Omega} V_{\mu}^+ \, d\nu_M \stackrel{\text{ii)}}{=} \int_B V_{\mu}^+ \, d\nu_M. \quad (4.4)$$

According to the representation (2.6) of Proposition 2.1 for the potential V_{μ} we have

$$\begin{aligned} I_{\mu} &= \int_B \left(\int_{\Omega} (k(\zeta, z) - k(\zeta, 0)) \, d\mu(z) \right)^+ \, d\nu_M(\zeta) \\ &\leq \int_B \int_{\Omega} (k(\zeta, z) - k(\zeta, 0))^+ \, d\mu(z) \, d\nu_M(\zeta) \\ &= \int_B \int_{\Omega} \left((k(\zeta, z) - k(\zeta, 0)) + (k(\zeta, 0) - k(\zeta, z))^+ \right) \, d\mu(z) \, d\nu_M(\zeta) \end{aligned}$$

Hence, by Fubini's theorem,

$$\begin{aligned} I_{\mu} &\leq \int_{\Omega} \left(\int_B (k(\zeta, z) - k(\zeta, 0)) \, d\nu_M(\zeta) \right. \\ &\quad \left. + \int_B (k(\zeta, 0) - k(\zeta, z))^+ \, d\nu_M(\zeta) \right) \, d\mu(z) \\ &\stackrel{(1.13)}{=} \int_{\Omega} \left(\int_B (k(\zeta, z) - k(\zeta, 0)) \, d\nu_M(\zeta) + Q_k^{\nu_M}(z) \right) \, d\mu(z). \quad (4.5) \end{aligned}$$

By condition ii) and representation (1.8) of Proposition 1.1 the function M can be represented in the form

$$M(z) = \int_B k(\zeta, z) d\nu_M(\zeta) + H(z) = U_k^{\nu_M}(z) + H(z), \quad z \in \Omega,$$

where $H \in \text{Harm}(\Omega)$. From this it follows that

$$(M(z) - M(0)) - (H(z) - H(0)) = \int_B (k(\zeta, z) - k(\zeta, 0)) d\nu_M(\zeta). \quad (4.6)$$

Substituting the left-hand side of (4.6) in (4.5), we get

$$I_\mu \stackrel{\text{iii)}}{\leq} \int_\Omega \left((M(z) - M(0)) - (H(z) - H(0)) + Q(z) \right) d\mu(z). \quad (4.7)$$

By Definition 2 of Arens–Singer (probability (2.3)) measures the right-hand side of (4.7) is equal to

$$\begin{aligned} \int_\Omega \left((M(z) + Q(z)) - H(z) \right) d\mu(z) - M(0) + H(0) \\ \stackrel{(2.2)}{=} \int_\Omega (M(z) + Q(z)) d\mu(z) - M(0) \end{aligned}$$

since $H \in \text{Harm}(\Omega)$. Now if we recall (4.3), (4.4), and (4.7), we get

$$\int_\Omega V_\mu d\nu_u \leq \int_\Omega (M(z) + Q(z)) d\mu(z) + (C' - M(0))$$

where the constant $C'' := C' - M(0)$ is independent of $\mu \in \mathcal{AS}^{\Omega_0}(\Omega) \cap \mathcal{M}_{\text{ac}}^+(\Omega)$. It means that second inequality in (2.19) is fulfilled *for every Arens–Singer measure* $\mu \in \mathcal{AS}^{\Omega_0}(\Omega) \cap \mathcal{M}_{\text{ac}}^+(\Omega)$ with the the function $M+Q \in L_{\text{loc}}^1(\Omega)$ instead of M , and with the constant C'' instead of C . Thus, by Theorem 2.3, for any function $\sigma \in C(\Omega)$ satisfying (1.11), there exists a function $h \in \text{Harm}(\Omega)$ such that

$$u(z) + h(z) \leq (M + Q)^{(\sigma)}(z) = (M^{(\sigma)} + Q^{(\sigma)})(z), \quad \forall z \in \Omega.$$

In other words, the function h is a harmonic minorant for the function (1.16).

This completes the proof of the Main Theorem. •

4.2 The proof of Corollary 1.1

In this subsection we use a some function $\sigma \in C(\Omega)$ satisfying (1.11).

The proof of (Z_{sc}). Let f_Λ be a holomorphic function with zero set $\Lambda = \{\lambda_n\}$, which exists by Weierstrass' theorem. Then the function $u = \log |f_\Lambda|$ is subharmonic with the Riesz measure n_Λ where n_Λ is the counting measure of sequence Λ (see (0.1)). At that rate the inequality (1.17) is exactly (1.14) with the Riesz measure $\nu_u = n_\Lambda$ of u . By assertion (Z) of the Main Theorem there exists a function $h \in \text{Harm}(\Omega)$ such that the inequality (1.15) holds. The domain Ω is simply connected. Therefore there exists a holomorphic function g on Ω such that $h = \text{Re } g$ on Ω . Thus there is a holomorphic function g such that

$$\log |f_\Lambda(z)| + \text{Re } g(z) = u(z) + h(z) \stackrel{(1.15)}{\leq} M^{(\sigma)}(z) + Q^{(\sigma)}(z), \quad \forall z \in \Omega. \quad (4.8)$$

In other words the function $f = f_\Lambda \exp g$ with $\text{Zero}_f = \text{Zero}_{f_\Lambda} = \Lambda$ satisfies the inequality $\log |f| \leq M^{(\sigma)} + Q^{(\sigma)}$ on Ω . This proves that Λ is a zero set for $\text{Hol}(\Omega; M^{(\sigma)} + Q^{(\sigma)})$. •

The proof of (S_{sc}). If Λ is zero subset for $\text{Hol}(\Omega; M)$, then there exists a holomorphic function $f \not\equiv 0$ on Ω such that $f(\Lambda) = 0$, and $f \in \text{Hol}(\Omega; M)$. Let f_Λ be a holomorphic function with zero set Λ . Evidently, we have a representation $f = f_\Lambda g$ where $g \not\equiv 0$ is a holomorphic function on Ω . The condition $f \in \text{Hol}(\Omega; M)$ implies an estimate

$$\log |f_\Lambda(z)| + \log |g(z)| \leq M(z) + C, \quad \forall z \in \Omega, \quad (4.9)$$

where C is a constant. We can rewrite (4.9) as

$$h(z) := \log |g(z)| - C \leq M(z) - \log |f_\Lambda(z)|, \quad \forall z \in \Omega.$$

In particular, it means that the difference $M - \log |f_\Lambda|$ admits a subharmonic minorant on Ω . Therefore, by statement (S) of the Main Theorem (with $\log |f_\Lambda|$ instead of u), the function $M^{(\sigma)} + Q^{(\sigma)} - \log |f_\Lambda|$ admits a harmonic minorant h on Ω . Since the domain Ω is simply connected, there exists a holomorphic function g on Ω such that $h = \text{Re } g$ on Ω . Thus we get (4.8). As before, this shows that Λ is a zero set for $\text{Hol}(\Omega; M^{(\sigma)} + Q^{(\sigma)})$. •

The proof of (M_{sc}). Let $f = g/q$ be a meromorphic function on Ω . If $g, q \in \text{Hol}(\Omega; M)$, then the subharmonic function $u := \max\{\log |g|, \log |q|\}$ satisfies

$$\max\{\log |g(z)|, \log |q(z)|\} = u(z) \leq M(z) + C, \quad \forall z \in \Omega, \quad (4.10)$$

where C is a constant. It is clear that there is a representation

$$f = \frac{g_1}{q_1} = \frac{g_1 l}{q_1 l}, \quad g = g_1 l, \quad q = q_1 l, \quad l \in \text{Hol}(\Omega), \quad l \not\equiv 0, \quad (4.11)$$

where g_1, q_1 are holomorphic functions without common zeros on Ω . Hence, in view of (4.10), we have

$$\max\{\log |g_1(z)|, \log |q_1(z)|\} + \log |l(z)| = u(z) \leq M(z) + C \quad (4.12)$$

for all $z \in \Omega$. Put

$$u_1 := \max\{\log |g_1|, \log |q_1|\}. \quad (4.13)$$

Under this notation we can rewrite (4.12) as $\log |l| - C \leq M - u_1$ on Ω . It means that the difference $M - u_1$ admits the subharmonic minorant $\log |l| - C$. Therefore, by assertion (S) of the Main Theorem, the function $M^{(\sigma)} + Q^{(\sigma)} - u_1$ admits a harmonic minorant. In other words there exists a holomorphic function s such that $u_1 + \operatorname{Re} s \leq M^{(\sigma)} + Q^{(\sigma)}$. Thus according to (4.13),

$$\max\{\log |g_1 e^s|, \log |q_1 e^s|\} \leq M^{(\sigma)} + Q^{(\sigma)}.$$

Hence if we put $g_0 = g_1 e^s$ and $q_0 = q_1 e^s$, then in view of (4.11) $f = g_0/q_0$, $g_0, q_0 \in \operatorname{Hol}(\Omega; M^{(\sigma)} + Q^{(\sigma)})$, and $\operatorname{Zero}_{g_0} \cap \operatorname{Zero}_{q_0} = \emptyset$ as desired. •

5 The proof of Theorem 1.1

5.1 The reduction

to an upper estimate of function (1.13)

Since M is bounded below at the origin, we have

$$\int_0^{1/2} \frac{\nu_M^{\operatorname{rad}}(t)}{t} dt < +\infty, \quad \nu_M(\{0\}) = 0.$$

It and the condition (1.20) imply that the subharmonic Bomash's kernel (1.6) from the item (\overline{B}_s) of Section 1 is suitable for the Riesz measure ν_M of M (see (1.7)). We use the subharmonic Bomash's kernel $\bar{b}_2(\zeta, z)$ in the role of subharmonic kernel k supported by $\mathbb{D} \setminus \{0\}$. In this case three conditions i)–iii) of the Main Theorem are fulfilled. It follows from Corollary 1.1 that all statements $(Z_{\mathbb{D}})$, $(S_{\mathbb{D}})$, and $(M_{\mathbb{D}})$ of Theorem 1.1 are fulfilled for the space $\operatorname{Hol}(\mathbb{D}; M^{(\sigma)} + Q^{(\sigma)})$ where

$$Q(z) := \int_{\mathbb{D}} (\bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, z))^+ d\nu_M(\zeta), \quad z \in \Omega. \quad (5.1)$$

By construction this function Q belongs to $L_{\operatorname{loc}}^1(\mathbb{D})$ since the function M is bounded below on every $D \Subset \mathbb{D}$.

Lemma 5.1. *There is an absolute constant a such that, for any $0 < \varepsilon < 1$,*

$$Q(z) \leq A_{M,\varepsilon}(z) - M(z) + C_\varepsilon \cdot b_M^{[\alpha']}(z), \quad a \leq |z| < 1, \quad (5.2)$$

where the functions $A_{M,\varepsilon}$ and $b_M^{[\alpha]}$ is defined by (1.19) and (1.21), $\alpha' > 0$ is an absolute constant, and C_ε depends only on ε .

We shall prove the important upper estimate (5.2) later by a few steps. Now we choose a function $\sigma \in C(\mathbb{D})$ satisfying (1.11) so that

$$\sigma(z) \leq \varepsilon(1 - |z|), \quad (A_{M,\varepsilon})^{(\sigma)}(z) \leq A_{M,\varepsilon}(z) + 1, \quad \forall z \in \mathbb{D}. \quad (5.3)$$

It is possible since the function $A_{M,\varepsilon}$ is continuous on \mathbb{D} if M is subharmonic. It follows from (5.3) and (5.2) that

$$\begin{aligned} M^{(\sigma)}(z) + Q^{(\sigma)}(z) &\leq A_{M,\varepsilon}(z) + (A_{M,\varepsilon}(z) - M(z) + C_\varepsilon \cdot b_M^{[\alpha']}(z))^{(\sigma)} \\ &\leq A_{M,\varepsilon}(z) + (A_{M,\varepsilon})^{(\sigma)}(z) - M^{(\sigma)}(z) + C_\varepsilon \cdot (b_M^{[\alpha']})^{(\sigma)}(z) \\ &\leq 2A_{M,\varepsilon}(z) - M(z) + C_\varepsilon \cdot b_M^{[\alpha'+1]}(z) + 1, \quad a < |z| < 1. \end{aligned}$$

Besides, the left-hand side is bounded below in $D(a)$. Thus the statements $(Z_{\mathbb{D}})$, $(S_{\mathbb{D}})$, $(M_{\mathbb{D}})$ hold for the space (1.22) with absolute constant $\alpha = \alpha' + 1$. •

5.2 The proof of Lemma 5.1

It is enough to estimate of integral (5.1) in the case $z = x \in [0, 1)$. First we investigate sizes of sets (see Examples, (1.3) and (B_0) , (\bar{B}_s) and (1.6))

$$D_1(x) := \{\zeta \in \mathbb{D} : \bar{b}_1(\zeta, 0) > \bar{b}_1(\zeta, x)\} = \{\zeta \in \mathbb{D} : \rho(\zeta, x) < |\zeta|\}, \quad (5.4.1)$$

$$\begin{aligned} D_2(x) &:= \{\zeta \in \mathbb{D} : \bar{b}_2(\zeta, 0) > \bar{b}_2(\zeta, x)\} \\ &\stackrel{(1.6b)}{=} \{\zeta \in \mathbb{D} : |\zeta|^2(2 - |\zeta|^2) > |\bar{B}_\zeta(x)||2 - \bar{B}_\zeta(x)|\}. \end{aligned} \quad (5.4.2)$$

The domain $D_2(x)$ determines the function Q from (5.1) since

$$Q(x) = \int_{D_2(x)} (\bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, x)) \, d\nu_M(\zeta), \quad x \in [0, 1). \quad (5.5)$$

Besides, the function $t \rightarrow t^2(2 - t^2)$ is strictly increasing on $[0, 1)$, and $|\bar{B}_\zeta(x)| < 1$ for $\zeta \in \mathbb{D}$. Therefore, if $\zeta \in D_2(x)$, then, by (5.4.2), we have $|\zeta|^2 > |\bar{B}_\zeta(x)|$, i. e., $|\zeta| > \rho(\zeta, x)$. Thus,

$$D_2(x) \subset D_1(x), \quad \forall x \in [0, 1). \quad (5.6)$$

The domain $D_1(x)$. We put

$$\zeta := te^{i\theta}, \quad 0 \leq t < 1, \quad \theta \in (-\pi, \pi]. \quad (5.7)$$

If $\rho(\zeta, x) \leq |\zeta|$, then

$$\left| \frac{te^{i\theta} - x}{1 - txe^{-i\theta}} \right|^2 = \frac{t^2 - 2tx \cos \theta + x^2}{1 - 2tx \cos \theta + t^2x^2} < t^2.$$

The last inequality is equivalent to

$$\cos \theta > \frac{x(1+t^2)}{2t}. \quad (5.8)$$

Hence $\cos \theta > x$ and

$$|\theta| < \frac{\pi}{2} \sqrt{1-x^2}. \quad (5.9)$$

Besides, inequality (5.8) implies $x(1+t^2) < 2t$. Hence, for $t = |\zeta|$ of (5.7), we obtain

$$1 > t > \frac{1 - \sqrt{1-x^2}}{x} \geq x - \sqrt{1-x^2}, \quad 0 \leq x < 1. \quad (5.10)$$

The inequalities (5.9) and (5.10) give the inclusion

$$D_1(x) \subset \left\{ \zeta = te^{i\theta} : x - \sqrt{1-x^2} < t < 1, \quad |\theta| < \frac{\pi}{2} \sqrt{1-x^2} \right\}. \quad (5.11)$$

But this inclusion is insufficient for good estimates of integral (5.5) by (5.6). It is important farther what, according to (5.6) and (5.11), the set $D_2(x)$ lies in the right half-plane for $x > 0$.

The domain $D_2(x)$. Let $\zeta \in D_2(x)$ the same as in (5.7).

Proposition 5.1. *Under the condition $9/10 \leq x < 1$ the Carleson box of relative size 6 with center at x (see (1.18))*

$$\square_6(x) = \left\{ \zeta = te^{i\theta} : x - 6(1-x) < t < 1, \quad |\theta| < 6(1-x) \right\} \quad (5.12)$$

includes the domain $D_2(x)$.

Proof. In view of (1.6b), we have

$$\log(|\zeta|(2 - |\zeta|^2)) > \log \frac{|\zeta - x| |2 - |\zeta|^2 - \bar{\zeta}x|}{|1 - \bar{\zeta}x|^2}$$

for $\zeta = te^{i\theta} \in D_2(x)$. Hence, under the notation (5.7),

$$\begin{aligned} t^2(2-t^2)^2 &> \frac{(t^2 - 2tx \cos \theta + x^2)((2-t^2)^2 - 2(2-t^2)tx \cos \theta + t^2x^2)}{(1 - 2tx \cos \theta + t^2x^2)^2} \\ &= \frac{((t-x)^2 + 4tx \sin^2(\theta/2))((2-t^2-tx)^2 + 4(2-t^2)tx \sin^2(\theta/2))}{((1-tx)^2 + 4tx \sin^2(\theta/2))^2}. \end{aligned} \quad (5.13)$$

In particular, $t > 0$. For convenience, we put

$$s := \sin^2 \frac{\theta}{2}. \quad (5.14)$$

Besides, $(2-t^2) \geq t^2(2-t^2)^2$ for $t \in [0, 1)$. Therefore, the inequality (5.13) implies

$$2-t^2 > \frac{((t-x)^2 + 4txs)((2-t^2-tx)^2 + 4(2-t^2)txs)}{((1-tx)^2 + 4txs)^2},$$

whence a direct calculation gives

$$\begin{aligned} s \cdot q(t, x) &:= s \cdot 4tx((2-t^2-tx)^2 + (t-x)^2(2-t^2) - 2(1-tx)^2(2-t^2)) \\ &< (1-tx)^4(2-t^2) - (t-x)^2(2-t^2-tx)^2 =: p(t, x), \end{aligned} \quad (5.15)$$

where the polynomials q and p are defined by first and second equalities respectively. We have the following factorial expansion for co-factor q after s :

$$q(t, x) = 8x^3t(1-t^2)^2 > 0, \quad t \in (0, 1), \quad x \in (0, 1).$$

It permits to get an upper estimate for s :

$$\begin{aligned} s < \frac{p(t, x)}{q(t, x)} &= \frac{(1-tx)^4(2-t^2) - (t-x)^2(2-t^2-tx)^2}{8tx^3(1-t^2)^2} \\ &= \frac{(1-t)^2(1+t^2)(-t^2 - x^4t^2 + 4x^3t + 2 - 4x^2)}{8x^3t(1-t^2)^2} \\ &= \frac{1}{8x^3t} (-t^2 - x^4t^2 + 4x^3t + 2 - 4x^2) \\ &= \frac{1}{8x^3t} (-(1+x^4)t^2 + (4x^3)t + 2(1-2x^2)) =: \frac{1}{8x^3t} g_x(t), \end{aligned} \quad (5.16)$$

where the quadratic trinomial g_x is defined by last equality. By definition (5.14), $s \geq 0$. Therefore the quadratic trinomial $g_x(t)$ is strictly positive for $\zeta = te^{i\theta} \in D_2(x)$, whence

$$\begin{aligned} t &> \frac{2x^3 - \sqrt{2}(1-x^2)}{1+x^4} = x - \left(x - \frac{2x^3 - \sqrt{2}(1-x^2)}{1+x^4} \right) \\ &= x - (1-x^2) \frac{\sqrt{2} + x(1-x^2)}{1+x^4} \geq x - 2(1-x^2) \geq x - 4(1-x). \end{aligned} \quad (5.17)$$

In particular,

$$\text{if } x \geq 9/10, \text{ then } t > 1/2. \quad (5.18)$$

Now we must find an upper estimate for s . Let us consider the inequality (5.16) again. The quadratic trinomial g_x attains its maximum at the point $t_x = 2x^3/(1+x^4)$ so that

$$\begin{aligned} g_x(t_x) &= -(1+x^4) \left(\frac{2x^3}{1+x^4} \right)^2 + 4x^3 \frac{2x^3}{1+x^4} + 2(1-2x^2) \\ &= \frac{2-4x^2+2x^4}{1+x^4} = \frac{2(1-x^2)^2}{1+x^4} \leq 2(1-x)^2 \max_{0 \leq x < 1} \frac{(1+x)^2}{1+x^4} \leq 8(1-x)^2 \end{aligned}$$

for each $x \in [0, 1)$. Hence, in view of (5.14) and (5.16), we get

$$\sin^2 \frac{\theta}{2} = s < \frac{1}{8x^3t} \cdot 8(1-x)^2 = \frac{1}{x^3t} \cdot (1-x)^2$$

whence, for $x \geq 9/10$, by (5.18) we obtain

$$|\theta| \leq \pi \sqrt{\frac{10^3}{9^3 \cdot (1/2)}} (1-x) < 6(1-x). \quad (5.19)$$

Thus, taking into account (5.17), we have the inclusion

$$D_2(x) \subset \{ \zeta = te^{i\theta} : x - 4(1-x) < t < 1, |\theta| < 6(1-x) \} \quad (5.20)$$

for all $9/10 \leq x < 1$. This completes the proof of Proposition 5.1. \bullet

The estimates of kernel of integral (5.5). Let $0 < \varepsilon < 1$. We put

$$\Delta_\varepsilon(x) := D(x, \varepsilon(1-x)). \quad (5.21)$$

Denote by

$$g_{\Delta_\varepsilon(x)}(\zeta, x) := \log \left| \frac{\varepsilon(1-x)}{\zeta-x} \right| \quad (5.22)$$

the Green's function for the disk $\Delta_\varepsilon(x)$ with the pole x .

Proposition 5.2. *If $9/10 \leq x < 1$, then the inequality*

$$(\bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, x))^+ \leq g_{\Delta_\varepsilon(x)}(\zeta, x) + \log \frac{30}{\varepsilon} \quad (5.23)$$

holds for all $\zeta \in \Delta_\varepsilon(x)$. Besides,

$$\frac{(1-|\zeta|)^2}{(1-x)^2} \geq 1-\varepsilon, \quad \forall \zeta \in \Delta_\varepsilon(x). \quad (5.24)$$

Proof. By (1.6b) and (5.22) we obtain the following representation

$$\bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, x) = g_{\Delta_\varepsilon(x)}(\zeta, x) + \log \frac{|\zeta|(2 - |\zeta|^2)|1 - \bar{\zeta}x|^2}{\varepsilon(1-x)|2 - |\zeta|^2 - \bar{\zeta}x|}. \quad (5.25)$$

Given $\zeta \in \mathbb{D}$ and $0 \leq x < 1$, we have $|\zeta|(2 - |\zeta|^2) \leq 2$,

$$|2 - |\zeta|^2 - \bar{\zeta}x| = |2 - \bar{\zeta}(\zeta + x)| \geq 2 - |\bar{\zeta}|(|\zeta| + x) \geq 1 - x. \quad (5.26)$$

Finally, for $0 < \varepsilon < 1$, $9/10 \leq x < 1$, under notation (5.7), we obtain

$$\begin{aligned} |1 - \bar{\zeta}x|^2 &= (1 - tx)^2 + 4tx \sin^2 \frac{\theta}{2} \leq (1 - (x - \varepsilon(1-x))x)^2 + 4tx \left(\frac{\varepsilon(1-x)}{x} \right)^2 \\ &\leq (1-x)^2((1+x+\varepsilon x)^2 + 4t\varepsilon^2/x) \leq (1-x)^2(9 + 4/x) \leq 15(1-x)^2. \end{aligned}$$

These tree estimates along with (5.25) give right away (5.23).

The lower estimate (5.24) is trivial. •

Proposition 5.3. *If $x \notin \Delta_\varepsilon(x)$, then*

$$(\bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, x))^+ \leq \frac{12(1 - |\zeta|)^2}{\varepsilon(1-x)^2}. \quad (5.27)$$

Proof. It follows from the representation (1.6a) that

$$\begin{aligned} (\bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, x))^+ &= \log^+ \left| \frac{1 - (1 - |\zeta|^2)^2}{1 - \left(\frac{1 - |\zeta|^2}{1 - \bar{\zeta}x} \right)^2} \right| \\ &= \log^+ \left| 1 + \frac{\left(\frac{1 - |\zeta|^2}{1 - \bar{\zeta}x} \right)^2 - (1 - |\zeta|^2)^2}{1 - \left(\frac{1 - |\zeta|^2}{1 - \bar{\zeta}x} \right)^2} \right| \leq \left| \frac{\left(\frac{1 - |\zeta|^2}{1 - \bar{\zeta}x} \right)^2 - (1 - |\zeta|^2)^2}{1 - \left(\frac{1 - |\zeta|^2}{1 - \bar{\zeta}x} \right)^2} \right| \\ &= (1 - |\zeta|^2)^2 \frac{|1 - (1 - \bar{\zeta}x)^2|}{|(1 - \bar{\zeta}x)^2 - (1 - |\zeta|^2)^2|} = (1 - |\zeta|^2)^2 \frac{(1 + |\zeta|)^2 x |2 - \bar{\zeta}x|}{|\zeta - x| |2 - |\zeta|^2 - \bar{\zeta}x|}. \end{aligned}$$

Hence, if we use (5.26), and the condition $|\zeta - x| \geq \varepsilon(1-x)$, then we obtain

$$(\bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, x))^+ \leq (1 - |\zeta|)^2 \frac{2^2 \cdot 3}{\varepsilon(1-x)(1-x)},$$

and the estimate (5.27) is proved. •

The upper estimate of (5.5). It follows from the inclusion (5.12) of Proposition 5.1 that

$$Q(x) \leq \int_{\square_6(x)} (\bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, x))^+ d\nu_M(\zeta)$$

$$\stackrel{(5.21)}{=} \left(\int_{\Delta_\varepsilon(x)} + \int_{\square_6(x) \setminus \Delta_\varepsilon(x)} \right) (\bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, x))^+ d\nu_M(\zeta) =: I_\varepsilon(x) + J(x)$$

for all $9/10 \leq x < 1$, where the integrals $I_\varepsilon(x)$ and $J(x)$ are defined by the last equality. For $I_\varepsilon(x)$, by estimate (5.23) of Proposition 5.2, we obtain the inequality

$$I_\varepsilon(x) := \int_{\Delta_\varepsilon(x)} (\bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, x))^+ d\nu_M(\zeta)$$

$$\leq \int_{\Delta_\varepsilon(x)} g_{\Delta_\varepsilon(x)}(\zeta, x) d\nu_M(\zeta) + \int_{\Delta_\varepsilon(x)} \log \frac{30}{\varepsilon} d\nu_M(\zeta)$$

$$\stackrel{(5.24)}{\leq} \int_{\Delta_\varepsilon(x)} g_{\Delta_\varepsilon(x)}(\zeta, x) d\nu_M(\zeta) + \frac{1}{1-\varepsilon} \log \frac{30}{\varepsilon} \cdot \int_{\Delta_\varepsilon(x)} \frac{(1-|\zeta|)^2}{(1-x)^2} d\nu_M(\zeta)$$

for all $9/10 \leq x < 1$. But by the Poisson–Jensen formula for the disk $\Delta_\varepsilon(x)$ we have

$$\int_{\Delta_\varepsilon(x)} g_{\Delta_\varepsilon(x)}(\zeta, x) d\nu_M(\zeta) = A_{M,\varepsilon}(x) - M(x).$$

Therefore,

$$I_\varepsilon(x) \leq A_{M,\varepsilon}(x) - M(x) + C'_\varepsilon \cdot \frac{1}{(1-x)^2} \int_{\Delta_\varepsilon(x)} (1-|\zeta|)^2 d\nu_M(\zeta) \quad (5.28)$$

for all $9/10 \leq x < 1$ where the constant $\frac{1}{1-\varepsilon} \log \frac{30}{\varepsilon} =: C'_\varepsilon$ depends only on ε .

For the integral $J(x)$, by estimate (5.27) of Proposition 5.3, we have

$$J(x) := \int_{\square_6(x) \setminus \Delta_\varepsilon(x)} (\bar{b}_2(\zeta, 0) - \bar{b}_2(\zeta, x))^+ d\nu_M(\zeta)$$

$$\leq \frac{12}{\varepsilon} \int_{\square_6(x) \setminus \Delta_\varepsilon(x)} \frac{(1-|\zeta|)^2}{(1-x)^2} d\nu_M(\zeta).$$

$$= C''_\varepsilon \cdot \frac{1}{(1-x)^2} \int_{\square_6(x) \setminus \Delta_\varepsilon(x)} (1-|\zeta|)^2 d\nu_M(\zeta) \quad (5.29)$$

where the constant $12/\varepsilon =: C''_\varepsilon$ also depends only on ε . The addition of (5.28) and (5.29) gives

$$I_\varepsilon(x) + J(x) \leq A_{M,\varepsilon}(x) - M(x) + C_\varepsilon b_M^{[6]}(x) \quad (5.30)$$

for all $9/10 \leq x < 1$ where the constant $C_\varepsilon := \max\{C'_\varepsilon, C''_\varepsilon\}$ depends only on ε . Thus Lemma 5.1 is proved with the constants $a = 9/10$ and $\alpha' = 6$. •

Remark. Since $\alpha = \alpha' + 1$ in subsection 5.1, it follows that we can put $\alpha = 7$ in the definition of space (1.22) in Theorem 1.1.

6 The proof of Theorem 0.1

We prove Theorem 0.1 as a corollary of Theorem 1.1.

By condition of Theorem 0.1 the function $M|_{[0,1]}$ is the increasing positive convex function of \log on $(0, 1)$. Therefore the function $M(z) = M(|z|)$ is continuous subharmonic function on \mathbb{D} , and there exists the positive left-hand derivative M'_- of $M(t)$, $t \in (0, 1)$. Besides, the function $tM'_-(t)$ is increasing on $(0, 1)$. In particular,

$$\int_r^{1-} (1-t) dM(t) = \int_r^{1-} (1-t)M'_-(t) dt \quad (6.1i)$$

$$\geq rM'_-(r) \int_r^{1-} \frac{1-t}{t} dt \geq rM'_-(r) \frac{1}{r} \frac{1}{2} (1-r)^2 = \frac{1}{2} M'_-(r)(1-r)^2 \quad (6.1r)$$

whence, according to the condition (0.4), we obtain

$$\lim_{r \rightarrow 1-} M'_-(r)(1-r)^2 = 0. \quad (6.2)$$

An easy calculation of Laplacian of M gives the expression

$$d\nu_M(z) = \frac{1}{2\pi} d\theta \otimes d(tM'_-(t)), \quad z := te^{i\theta}, \quad 0 \leq t < 1, \quad (6.3)$$

for the density of Riesz measure ν_M of $M \in SH(\mathbb{D})$ (in the sense of distribution theory).

Suppose that α is the absolute constant for the space (1.22) in Theorem 1.1. In our case we have ($z = re^{i\theta}$)

$$\begin{aligned} b_M^{[\alpha]}(z) &\stackrel{(1.21)}{=} b_M^{[\alpha]}(r) \\ &\stackrel{(1.18), (6.3)}{=} \frac{1}{(1-r)^2} \frac{1}{2\pi} \int_{-\alpha(1-r)}^{\alpha(1-r)} \left(\int_{r-\alpha(1-r)}^{1-} (1-t)^2 d(tM'_-(t)) \right) d\theta \\ &= \frac{\alpha}{\pi} \frac{1}{1-r} \int_{r-\alpha(1-r)}^{1-} (1-t)^2 d(tM'_-(t)). \end{aligned}$$

Hence integration by parts gives (taking into account (6.2) and the positivity

of the function $tM'_-(t)$

$$\begin{aligned}
b_M^{[\alpha]}(z) &\leq \frac{\alpha}{\pi} \frac{1}{1-r} 2 \int_{r-\alpha(1-r)}^{1-} tM'_-(t)(1-t) dt \\
&\frac{2\alpha}{\pi} \frac{1}{1-r} \int_{r-\alpha(1-r)}^{1-} M'_-(t)(1-t) dt \stackrel{(6.1i)}{=} \frac{2\alpha}{\pi} \frac{1}{1-r} \int_{r-\alpha(1-r)}^{1-} (1-t) dM(t) \\
&\stackrel{(0.4)}{\leq} \frac{2\alpha}{\pi} \frac{1}{1-r} \cdot C(1 - (r - \alpha(1-r))) = \frac{2\alpha}{\pi} C_M(1 + \alpha),
\end{aligned}$$

where a constant C_M is dictated by condition (0.4). In particular, the condition (1.20) is fulfilled. Thus, under the conditions of Theorem 0.1 the space $\text{Hol}(\mathbb{D}; 2A_{M,\varepsilon} - M + C_\varepsilon \cdot b_M^{[\alpha]})$ from (1.22) *coincides with the space* $\text{Hol}(\mathbb{D}, 2A_{M,\varepsilon} - M)$ for every $0 < \varepsilon < 1$.

Let us fix a number $\varepsilon \in (0, 1)$. The function M is increasing and radial. Therefore,

$$A_{M,\varepsilon}(z) \leq M(r + \varepsilon(1-r)), \quad r = |z|. \quad (6.4)$$

By the mean value theorem there exists a point r' such that $r \leq r' \leq r + \varepsilon(1-r)$ and

$$\begin{aligned}
M(r + \varepsilon(1-r)) - M(r) &\leq M'_-(r') \cdot \varepsilon(1-r) = M'_-(r')(1-r') \cdot \varepsilon \frac{1-r}{1-r'} \\
&\leq \frac{\varepsilon}{1-\varepsilon} M'_-(r')(1-r') \stackrel{(6.1)}{\leq} \frac{\varepsilon}{1-\varepsilon} \cdot \frac{2}{1-r'} \int_{r'}^{1-} (1-t) dM(t) \stackrel{(0.4)}{\leq} \frac{2\varepsilon}{1-\varepsilon} \cdot C_M,
\end{aligned}$$

where C_M is a constant which exists according to the condition (0.4). Hence, in view of (6.4), we obtain

$$2A_{M,\varepsilon}(z) - M(z) \leq 2M(r) + 2\frac{2\varepsilon}{1-\varepsilon} \cdot C_M - M(r) = M(r) + C, \quad r = |z|,$$

where $C = 4C_M\varepsilon/(1-\varepsilon)$ is a constant. It means that the space $\text{Hol}(\mathbb{D}, 2A_{M,\varepsilon} - M)$ *coincides with the space* $\text{Hol}(\mathbb{D}, M)$. Hence Theorem 0.1 is a special case of Theorem 1.1. •

7 The case of the uniform Bergman spaces

For $V \in \mathcal{P}_{AS}(\mathbb{D})$, we put

$$\widehat{\kappa}(V) := \int_0^{1-} \left(\frac{1}{2\pi} \int_0^{2\pi} V(te^{i\theta}) d\theta \right) \frac{dt}{(1-t)^2}. \quad (7.1)$$

7.1 Weak analogs of Korenblum–Seip’s conditions

Theorem 7.1. *Let $\Lambda = \{\lambda_k\}$, $k = 1, 2, \dots$, $0 \notin \Lambda$, be a sequence of points in \mathbb{D} , and $0 \leq p < +\infty$. The following four statements are equivalent:*

[Λ] *the sequence Λ is a zero set for the space A^{-p} ;*

[G] *there exists a constant $a < 1$ such that*

$$\sup_{D \in \mathcal{U}_0^d(\mathbb{D}; D(a))} \left(\sum_k g_D(\lambda_k, 0) - p \widehat{\kappa}(g_D(\cdot, 0)) \right) < \infty;$$

[J] $\sup_{V \in \mathcal{P}_J(\mathbb{D})} \left(\sum_k V(\lambda_k) - p \widehat{\kappa}(V) \right) < \infty;$

[AS] $\sup_{V \in \mathcal{P}_{AS}(\mathbb{D})} \left(\sum_k V(\lambda_k) - p \widehat{\kappa}(V) \right) < \infty.$

Evidently, the equivalence [Λ] \iff [G] coincides with the equivalence (i) \iff (ii) of Theorem 0.2.

Proof. Here we put

$$M_p(z) := p \log \frac{1}{1 - |z|}, \quad z \in \mathbb{D}. \quad (7.2)$$

This function is radial and positive. Besides, the restriction $M \big|_{(0,1)}$ is the increasing continuous convex function of \log on $(0, 1)$, and

$$dM_p(t) = \frac{1}{1-t} dt, \quad d(tM_p'(t)) = \frac{1}{(1-t)^2} dt. \quad (7.3)$$

In particular, condition (0.4) holds, and M_p is subharmonic on \mathbb{D} with the Riesz measure

$$d\nu_{M_p}(z) \stackrel{(6.3)}{=} \frac{p}{2\pi} d\theta \otimes \frac{dt}{(1-t)^2}, \quad z = te^{i\theta} \in \mathbb{D}, \quad 0 \leq r < 1. \quad (7.4)$$

By definition (0.3) we have $\text{Hol}(\mathbb{D}, M_p) = A^{-p}$. In this case condition [G] coincides with condition (0.5) of Theorem 0.1 with a constant C independent of D . By item (Z_r) of Theorem 0.1 condition [G] implies [Λ] and vice versa.

The evident inclusions of the class of all extended Green functions $g_D(\cdot, 0)$, $D \Subset \mathbb{D}$, into $\mathcal{P}_J(\mathbb{D}) \subset \mathcal{P}_{AS}(\mathbb{D})$ give the implications [AS] \implies [J] \implies [G].

For the proof of implication [Λ] \implies [AS] we use Proposition 2.6.

Let $f_\Lambda \in A^{-p}$ be a function with zero set Λ . It means that the function $\equiv 0$ is harmonic minorant for the difference $M_p - \log |f_\Lambda|$. Therefore, by Proposition 2.6, the inequalities (2.16) hold for all for all Arens–Singer functions V on \mathbb{D} with $u = \log |f_\Lambda|$, $\nu_u = n_\Lambda$, and $\nu_M = \nu_{M_p}$. In view of (7.4), the system of inequalities (2.16) is equivalent to [AS]. \bullet

7.2 Variants of Luecking's condition

Theorem 7.2. *Let $\Lambda = \{\lambda_k\}$, $k = 1, 2, \dots$, $0 \notin \Lambda$, be a sequence of points in \mathbb{D} and $0 \leq p < +\infty$. The following four statements are equivalent:*

[Λ] *The sequence Λ is a zero set for the space A^{-p} ;*

[GL] *there exists a constant $a < 1$ such that*

$$\sup_{D \in \mathcal{U}_0^d(\mathbb{D}; D(a))} \left(\sum_k (1 - |\lambda_k|^2)^2 \left(\frac{1}{\pi} \int_{\mathbb{D}} \frac{g_D(\zeta, 0) dm(\zeta)}{|1 - \lambda_k \bar{\zeta}|^4} \right) - p \widehat{\kappa}(g_D(\cdot, 0)) \right) < \infty;$$

$$[\text{JL}] \sup_{V \in \mathcal{P}_J(\mathbb{D})} \left(\sum_k (1 - |\lambda_k|^2)^2 \left(\frac{1}{\pi} \int_{\mathbb{D}} \frac{V(\zeta) dm(\zeta)}{|1 - \lambda_k \bar{\zeta}|^4} \right) - p \widehat{\kappa}(V) \right) < \infty;$$

$$[\text{ASL}] \sup_{V \in \mathcal{P}_{AS}(\mathbb{D})} \left(\sum_k (1 - |\lambda_k|^2)^2 \left(\frac{1}{\pi} \int_{\mathbb{D}} \frac{V(\zeta) dm(\zeta)}{|1 - \lambda_k \bar{\zeta}|^4} \right) - p \widehat{\kappa}(V) \right) < \infty.$$

Evidently, the equivalence [Λ] \iff [GL] coincides with the equivalence (i) \iff (iii) of Theorem 0.2.

Scheme of the proof. The evident inclusions of the class of all extended Green functions $g_D(\cdot, 0)$, $D \in \mathbb{D}$, into $\mathcal{P}_J(\mathbb{D}) \subset \mathcal{P}_{AS}(\mathbb{D})$ give the implications [ASL] \implies [JL] \implies [GL].

Put

$$K_\Lambda(z) = \frac{|z|^2}{2} \sum_{\lambda_k \in \Lambda} \frac{(1 - |\lambda_k|^2)^2}{|1 - \lambda_k \bar{z}|^2}. \quad (7.5)$$

The function K_Λ is a subharmonic continuous function on \mathbb{D} . Easy calculations give

$$\begin{aligned} \Delta K_\Lambda(z) &= 4 \frac{\partial^2}{\partial z \partial \bar{z}} K_\Lambda(z) \\ &= 2 \sum_k (1 - |\lambda_k|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}} \frac{z \bar{z}}{(1 - \lambda_k \bar{z})(1 - \bar{\lambda}_k z)} \\ &= 2 \sum_k (1 - |\lambda_k|^2)^2 \frac{1}{|1 - \lambda_k \bar{z}|^4}. \end{aligned}$$

Hence we get

$$d\nu_{K_\Lambda}(z) = \frac{1}{\pi} \sum_k \frac{(1 - |\lambda_k|^2)^2}{|1 - \lambda_k \bar{z}|^4} dm(z). \quad (7.6)$$

By Luecking's criterion [9, Theorem A, (b)] a sequence $\Lambda = \{\lambda_k\}$ is zero set for A^{-p} if and only if for M_p from (7.2), the function $M_p - K_\Lambda$ admits a harmonic minorant.

Let $M = M_p$, $\nu_M = \nu_{M_p}$ (see (7.4)), and $u = K_\Lambda$. It follows from Proposition 2.6 the same way as in the proof of implication $[\Lambda] \implies [\text{AS}]$ of previous Theorem that the implication $[\Lambda] \implies [\text{ASL}]$ holds.

For the proof of implication $[\text{GL}] \implies [\Lambda]$ we can use subharmonic analogs of Corollary 1.1 (item $[(Z_{\text{sc}})]$), Theorem 1.1 (item $[(Z_{\mathbb{D}})]$), and Theorem 0.1 (item $[(Z_r)]$) with the measure ν_{K_Λ} in place of measure n_Λ . In particular, the last is the statement

[SH_r] *Let the function M be the same as in Theorem 0.1, and K be a subharmonic continuous function on \mathbb{D} with Riesz measure ν_K . The function $M - K$ admits a harmonic minorant if and only if there are constants $a < 1$ and C such that the inequality*

$$\int_{\mathbb{D}} g_D(\zeta, 0) d\nu_K(\zeta) \leq \int_0^{1^-} \left(\frac{1}{2\pi} \int_0^{2\pi} g_D(te^\theta, 0) d\theta \right) d(tM'_-(t)) + C$$

holds for each domain $D \in \mathcal{U}_0^d(\mathbb{D}; D(a))$.

The system of inequalities in the statement [SH_r] is equivalent to the statement [GL] with K_Λ and M_p from (7.5) and (7.2) in place of K and M respectively according to (7.6) and (7.3). It proves the equivalence $[\text{GL}] \iff [\Lambda]$. •

Remarks. 1. The summands in first sums of [GL], [JL], and [ASL] are values at points λ_k of the Berezin transforms (see [8, Ch. 2]) of Green, Jensen and Arens–Singer functions respectively.

2. In Theorems 7.1 and 7.2 in items [G] and [GL] we can suppose that the domain $D \subset \mathbb{D}$ has final number of points of contact with circumference $\partial\mathbb{D}$ since the integral (7.1) with the integrand $V = g_D(\cdot, 0)$ is a finite number.

More detailed statement of this work will be submit in the journal “Matematicheskii sbornik” in Russian (the translation from Russian into English is “Sbornik: Mathematics”).

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