Distribution of Zero Subsequences for Bernstein class and Criteria of Completeness

for exponential system on a segment accurate within one or two exponential functions

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Outline

Spaces $C(I_d)$, and $L^p(I_d)$

- Denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of natural, real and complex numbers respectively; $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$, $\mathbb{C}_\pm := \mathbb{C} \setminus \mathbb{R}$.
- Let $I_d \subset \mathbb{R}$ be a segment of length d.
- We denote by $C(I_d)$ the space of continuous functions $f\colon I_d \to \mathbb{C}$ on I_d with sup-norm $\|f\|_\infty := \sup\{|f(x)|: x \in I_d\}$.
- and by $L^p(I_d)$ the space of functions f with finite norm

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Bernstein space B_σ^∞

For $\sigma \in (0, +\infty)$, denote by B_σ^∞ *the Bernstein space* (of type σ) of all entire (holomorphic on $\mathbb{C}$) functions f of exponential type $\leqslant \sigma$ bounded on $\mathbb{R}$, i. e.

$$\log |f(z)| \leqslant \sigma |\mathrm{Im}\, z| + c_f, \quad z \in \mathbb{C}, \tag{1.1}$$

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Zero subsequences

- Let
$$\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$$
be a point sequence on $\mathbb{C}$ without limit points in $\mathbb{C}$, and all points λ_k are pairwise different (*for simplicity*).
- The sequence Λ is a *zero subsequence* (non-uniqueness sequence) for B_σ^∞ iff there exists a nonzero function $f_\Lambda \in B_\sigma^\infty$ such that f_Λ vanish on Λ, i. e. $f_\Lambda(\lambda_k) = 0 \ \forall k \in \mathbb{N}$.

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Exponential systems.
Completeness

The *exponential system*

$$\mathrm{Exp}^{i\Lambda} := \left\{ e^{i\lambda_k x} \right\}_{k\in\mathbb{N}}, \quad x \in I_d,$$

is *complete* in $C(I_d)$ (in $L^p(I_d)$ respectively) iff the closure of its linear span coincides with $C(I_d)$ ($L^p(I_d)$ respectively).

Problems

We solve following two problems.

Complete description of zero subsequences for B_{σ}^{∞}.

Criteria of completeness of system $\mathrm{Exp}^{i\Lambda}$ in $C(I_d)$ or $L^p(I_d)$ accurate within one or two exponential functions.

It was many times noticed that obtaining of criteria of completeness for exponential systems $\mathrm{Exp}^{i\Lambda}$ in $C(I_d)$ or $L^p(I_d)$ purely in terms Λ and d is an exclusively complicated problem. See, for example,

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Poisson integral.
Properties

- For $z \in \mathbb{C}_\pm := \mathbb{C} \setminus \mathbb{R}$, *the Poisson integral* $\mathrm{P}_{\mathbb{C}_\pm}\varphi$ of function $\varphi \in L^1(\mathbb{R})$ is defined as

$$(\mathrm{P}_{\mathbb{C}_\pm}\varphi)(z) := \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{|\operatorname{Im} z|}{(t - \operatorname{Re} z)^2 + (\operatorname{Im} z)^2}\,\varphi(t)\,dt$$
$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} \left|\operatorname{Im}\frac{1}{t-z}\right| \varphi(t)\,dt, \qquad (1.2)$$

- But for $x \in \mathbb{R}_*$ we set

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The Poisson integral from (1.2) is harmonic function on $\mathbb{C}_{\pm}$. If $\varphi \in C(\mathbb{R}_*)$, then the Poisson transform $\mathrm{P}_{\mathbb{C}_{\pm}}\,\varphi$ is also continuous function on $\mathbb{C} \setminus \{0\}$.

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Hilbert transform

(direct). See [2].

For a function

$$\varphi \in L^1(\mathbb{R}_*), \quad \varphi\colon \mathbb{R}_* \to \mathbb{R},$$

the *direct Hilbert transform* H is defined as a rule by integral

$$(\mathrm{H}\,\varphi)(x) := \frac{1}{\pi} \int_{\mathbb{R}_*} \frac{\varphi(t)}{x-t}\,dt, \quad x \in \mathbb{R}_*,$$

where the strikethrough of integral

$$\int_{\mathbb{R}_*} := \lim_{0<\varepsilon\to 0} \int_{\mathbb{R}_*\setminus(x-\varepsilon,x+\varepsilon)}, \quad x \neq 0,$$

means the principal integral value in the sense of Cauchy. Such function $\mathrm{H}\,\varphi$ is good defined almost everywhere on $\mathbb{R}$.

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Hilbert transform

(inverse). See [2].

The *inverse Hilbert transform* differs only by sign

$$(\mathrm{H}^{-1}\varphi)(x) := \frac{1}{\pi}\int_{\mathbb{R}_*} \frac{\varphi(t)}{t-x}\,dt = -(\mathrm{H}\,\varphi)(x), \quad x \in \mathbb{R}_*.$$

Classes of test functions $R\mathcal{P}_0^m$, $2 \leqslant m \leqslant \infty$.

This class $R\mathcal{P}_0^m$, $2 \leqslant m \leqslant \infty$, will consist of all *positive functions* $\varphi\colon \mathbb{R}_* \to [0, +\infty)$ from the class $C^m(\mathbb{R}_*)$ of m times continuously differentiable functions on $\mathbb{R}_*$ with

- *a finiteness condition* $\varphi(x) \equiv 0, \quad |x| \geqslant R_\varphi > 0$, where $R_\varphi > 0$ is a constant;
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- *a conjugate condition of positivity*

$$(-\mathrm{H}\,\varphi)'(x) := \frac{1}{\pi} \int_{\mathbb{R}_*} \frac{\varphi(t) - \varphi(x)}{(t-x)^2}\, dt \geqslant 0, \; x \in \mathbb{R}_*, \qquad (2.1)$$

i. e. condition of increase for the inverse Hilbert transform $-\mathrm{H}\,\varphi$ separately on the negative semiaxis $(-\infty, 0)$ and on the positive semiaxis $(0, +\infty)$.

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It is easy show, that Hilbert transform and the differentiation operator commute, i. e.

$$\frac{d}{dx}(-\mathrm{H}\,\varphi) \equiv -\mathrm{H}\,\frac{d}{dx}\varphi,$$

for $\varphi \in L^p\bigl(\mathbb{R} \setminus \{0\}\bigr)$, $p \geqslant 1$, and for Schwartz distributions φ.
See

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Besides, the left-hand member of (2.1) can be rewritten as

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Main Theorem
(on zero subsequence for Bernstein space).

Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \not\ni 0$ be a point sequence on $\mathbb{C}$, and $\sigma \in (0, +\infty)$. The following assertions are equivalent:

1) Λ is zero subsequence for Bernstein space B^{∞}_{σ};

2) for the some (for any) $m \in (\mathbb{N} \setminus \{1\}) \cup \infty$ the condition

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Any shifts of any final number of points from Λ do not change his property as zero subsequence for B_σ^∞. Therefore, without loss of generality, we can consider only cases $0 \notin \Lambda$.

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Any shifts of any final number of points from Λ do not change his property as zero subsequence for B_σ^∞. Therefore, without loss of generality, we can consider only cases $0 \notin \Lambda$.

Remark 3

Conditions (3.1) and (3.2) are similar to definition of order relation on sets of real-valued Radon measures or on distributions. Indeed, if ν and μ are two such measures or two distributions on $\mathbb{C}$, then $\nu \leqslant \mu$ iff

$$\sup_{\varphi \in \left(C_0^\infty(\mathbb{C})\right)^+} \big(\nu(\varphi) - \mu(\varphi)\big) \leqslant 0, \tag{3.3}$$

where $(C_0^\infty(\mathbb{C}))^+$ is the class of infinitely differentiable finite positive functions on $\mathbb{C}$.

Remark 3

(continuation)

On the other hand, each point sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ defines a *counting measure*

$$\nu_\Lambda(S) := \sum_{\lambda_k \in S} 1, \quad S \subset \mathbb{C}.$$

The measure μ_σ with density $d\mu_\sigma(x) := \dfrac{\sigma}{\pi}\, dx, \in \mathbb{R}$, on $\mathbb{R}$ is the Riesz measure of subharmonic function $M_\sigma(z) := \sigma |\mathrm{Im}\, z|$, $z \in \mathbb{C}$, $\mu_\sigma := \frac{1}{2\pi} \Delta M_\sigma \geqslant 0$, where Δ is the Laplace operator in the sense of the Schwartz distribution theory.

Remark 3

(continuation)

The function M_σ is located in the right-hand member of the inequality (1.1) and defines the Bernstein space B_σ^∞. In these notations the conditions (3.1) and (3.2) it is possible to write in the form

$$\sup_{\varphi \in R\mathcal{P}_0^m} \bigl(\nu_\Lambda(\mathrm{P}_{\mathbb{C}_\pm}\varphi) - \mu_\sigma(\varphi)\bigr) < +\infty.$$

This record is useful for comparison to (3.3).

On the completeness of exponential systems
in $C(I_d)$, and $L^p(I_d)$

- A system of vectors of vector topological space is complete in this space iff the closure of its linear span coincides with this space. Otherwise the system is incomplete.
- Denote by $I_d \subset \mathbb{R}$ an arbitrary segment $[a, b]$ of length $d = b - a$.
- For spaces on a segment I_d we consider traditionally exponential systems with sequence of exponents

$$i\Lambda := \{i\lambda_k\}_{k\in\mathbb{N}}, \quad \mathrm{Exp}^{i\Lambda} = \{e^{i\lambda x} : \lambda \in \Lambda,\ x \in I_d\}.$$

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Theorem on completeness of exponential systems

in $C(I_d)$ and $L^p(I_d)$ accurate within one or two exponential functions

If a point sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \not\ni 0$ satisfies to the condition

$$S_d^m(i\Lambda) := \sup_{\varphi\in R\mathcal{P}_0^m} \left(\sum_{k\in\mathbb{N}} (\mathrm{P}_{\mathbb{C}_\pm}\varphi)(\lambda_k) - \frac{d}{\pi}\int_{-\infty}^{+\infty} \varphi(x)\,dx \right) = +\infty,$$

then the system $\mathrm{Exp}^{i\Lambda}$ is complete in $C(I_d)$ and $L^p(I_d)$, $p \geqslant 1$.

Inversely, if $S_d^m(i\Lambda) < +\infty$, then, for any pair different points $\{\lambda', \lambda''\} \subset \Lambda$, the system $\mathrm{Exp}^{i\Lambda\setminus\{i\lambda'\}}$ is incomplete in $C(I_d)$ and $L^p(I_d)$ for $p \geqslant 2$, and the system $\mathrm{Exp}^{i\Lambda\setminus\{i\lambda', i\lambda''\}}$ is incomplete in $L^p(I_d)$ for $1 \leqslant p < 2$.

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Examples

Remark 4

From our Theorem on completeness of exponential system we can obtain all basic old results in this direction for spaces $C(I_d)$ and $L^p(I_d)$ (for example, the Berling–Malliavin Theorem on radius of completeness), and also new results.

Let's prove very briefly and quickly one old result from

[4] L. Schwartz, *Approximation d'une fonction quelconque par des sommes d'exponentielles imaginaires*, Ann. Fac. Sci. Toulouse, (1943), 111–176.

[5] R. M. Redheffer, *Completeness of sets of complex exponential*, Adv. in Math., **24** (1977), 1–62.

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Theorem ([4], [5, Theorem 41])

If a point sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ *satisfies to two conditions*

$$0 < \alpha \leqslant |\arg \lambda_k| \leqslant \pi - \alpha,\ k \in \mathbb{N}, \quad \sum_{k\in\mathbb{N}} \left|\mathrm{Im}\,\frac{1}{\lambda_k}\right| < +\infty\,, \qquad (4.1)$$

then the system $\mathrm{Exp}^{i\Lambda}$ *is incomplete in any* $C(I_d)$ *and* $L^p(I_d)$.

Proof.

Without loss generality we can consider, that for arbitrary ε the condition

$$\sum_{k\in\mathbb{N}} \left|\operatorname{Im} \frac{1}{\lambda_k}\right| < \varepsilon \tag{4.2}$$

is fulfilled instead of second condition from (4.1). For this purpose it is enough to shift finite number of points. It does not change the property of (in-)completeness (see Remark 2). First condition from (4.1) implies the estimate $|\operatorname{Im}\lambda_k| \geqslant |\lambda_k| \sin\alpha$. This estimate give the following estimate for the Poisson kernel at every point λ_k:

$$\frac{1}{\pi}\frac{|\operatorname{Im}\lambda_k|}{(t-\operatorname{Re}\lambda_k)^2+(\operatorname{Im}\lambda_k)^2} \leqslant \frac{1}{\pi\sin^2\alpha}\frac{|\operatorname{Im}\lambda_k|}{|\lambda_k|^2} = \frac{1}{\pi\sin^2\alpha}\left|\operatorname{Im}\frac{1}{\lambda_k}\right|.$$

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⇓

Hence in view of (4.2) for arbitrary $\varphi \in R\mathcal{P}_0^m$ we have

$$\sum_{k\in\mathbb{N}}(\mathsf{P}_{\mathbb{C}_\pm}\,\varphi)(\lambda_k) := \sum_{k\in\mathbb{N}} \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{|\mathrm{Im}\,\lambda_k|}{(t-\mathrm{Re}\,\lambda_k)^2+(\mathrm{Im}\,\lambda_k)^2}\,\varphi(t)\,dt$$

$$\leqslant \sum_{k\in\mathbb{N}} \frac{1}{\pi\sin^2\alpha}\left|\mathrm{Im}\,\frac{1}{\lambda_k}\right|\cdot\int_{-\infty}^{+\infty}\varphi(t)\,dt \leqslant \frac{\varepsilon}{\pi\sin^2\alpha}\cdot\int_{-\infty}^{+\infty}\varphi(x)\,dx.$$

If we choose ε sufficiently small, then $\dfrac{\varepsilon}{\pi\sin^2\alpha} \leqslant \dfrac{d'}{\pi} < \dfrac{d}{\pi}$, and the condition $S^m_{d'}(i\Lambda) \leqslant 0$ is fulfilled. By second part of Theorem on completeness, the exponential system $\mathrm{Exp}^{i\Lambda}$ is incomplete in spaces $C(I_{d'})$ and $L^p(I_{d'})$ accurate within one or two functions. Thereby, the system $\mathrm{Exp}^{i\Lambda}$ is incomplete in spaces $C(I_d)$ and in $L^p(I_d)$ for any $d > 0$. The Theorem is proved. □

Let's remind the Beurling–Malliavin Theorem on the radius of completeness in the Redheffer's interpretation.

Beurling–Malliavin Theorem ([5, Theorem 77], [2])

Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$*. If there exists a number* $c > 0$ *and a sequence* $\{n_k\}_{k\in\mathbb{N}}$ *of distinct integers such that the series*

$$\sum_{k\in\mathbb{N}} \left| \frac{1}{\lambda_k} - \frac{c}{\pi n_k} \right| \tag{4.3}$$

converges, then the system $\operatorname{Exp}^{i\Lambda}$ *is incomplete in* $C(I_d)$ *and* $L^p(I_d)$ *for any* $d > c$. *Inversely, if the series* (4.3) *diverges for every sequence* $\{n_k\}_{k\in\mathbb{N}}$ *of distinct integers, then the system* $\operatorname{Exp}^{i\Lambda}$ *is complete in* $C(I_d)$ *and* $L^p(I_d)$ *for any* $d < c$.

On the basis of Theorem on completeness of exponential systems it is possible to give a new proof of this Theorem. But it

Theorem

Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ *and* $\Gamma = \{\gamma_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ *are two point sequences on* $\mathbb{C}$ *without limit points in* $\mathbb{C}$ *such that all points* λ_k *and all points* γ_k *are pairwise different. Suppose that there is* $h \in \mathbb{R}$ *such that* $\operatorname{Im}\lambda_k \neq -h$ *and* $\operatorname{Im}\gamma_k \neq -h$ *for all* $k \in \mathbb{N}$, *and*

$$\sup_{t\in\mathbb{R}} \sum_{k\in\mathbb{N}} \left(\frac{|\operatorname{Im}\lambda_k + h|}{(t-\operatorname{Re}\lambda_k)^2 + (\operatorname{Im}\lambda_k + h)^2} - \frac{|\operatorname{Im}\gamma_k + h|}{(t-\operatorname{Re}\gamma_k)^2 + (\operatorname{Im}\gamma_k + h)^2} \right) < +\infty.$$

If the sequence Γ *is a zero sequence for* B_σ^∞ , *then the sequence* Λ *is also a zero sequence for* B_σ^∞.
If the system $\operatorname{Exp}^{i\Lambda}$ *is complete in one of spaces* $C(I_d)$ *or* $L^p(I_d)$, $p \geqslant 1$, *then for any* $\gamma', \gamma'' \notin \Gamma$, $\gamma' \neq \gamma''$ *the system* $\operatorname{Exp}^{i\Gamma\cup\{i\gamma'\}}$ *is complete in* $C(I_d)$ *and in* $L^p(I_d)$ *for* $p \geqslant 2$, *and the system* $\operatorname{Exp}^{i\Gamma\cup\{i\gamma', i\gamma''\}}$ *is complete in* $L^p(I_d)$ *for* $1 \leqslant p < 2$.

Corollary

Let $\Lambda \cap \mathbb{R} = \varnothing$ *and* $\Gamma \cap \mathbb{R} = \varnothing$*. If*

$$\limsup_{t \to \pm\infty} \sum_{k \in \mathbb{N}} \left| \operatorname{Im} \frac{\lambda_k - \gamma_k}{(t - \lambda_k)(t - \gamma_k)} \right| < +\infty,$$

then Λ *and* Γ *can be zero sequences for* B_σ^∞ *only simultaneously.*
Besides, $|\operatorname{exc} i\Lambda - \operatorname{exc} i\Gamma| \leqslant 1$ *for* $C(I_d)$ *and* $L^p(I_d)$ *with* $p \geqslant 2$*, and* $|\operatorname{exc} i\Lambda - \operatorname{exc} i\Gamma| \leqslant 2$ *for* $L^p(I_d)$ *with* $1 \leqslant p < 2$*.*