Criteria of zero subsequences for spaces of entire functions of exponential type with restrictions of their growth on real axis

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Denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of natural, real, and complex numbers; $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$.

Let M be a subharmonic function on $\mathbb{C}$ such that M is harmonic on $\mathbb{C}\setminus\mathbb{R}$, $M(z) \leq O\bigl(|z|\bigr)$, $z \to \infty$, $M \in C^1\bigl(\{z \in \mathbb{C} \colon \Re z \geq 0\}\bigr)$ and $\in C^1\bigl(\{z \in \mathbb{C} \colon \Re z \leq 0\}\bigr)$, and M satisfies the condition $\sup\limits_{|z-w|\leq 1} M(w) \leq M(z) + C_M$, $z \in \mathbb{C}$, where C_M is a constant. We set

$$\nu_M(t) := \frac{1}{2\pi}\int_0^t \lim_{0<y\to 0} \frac{M(x+iy)+M(x-iy)}{y}\,dx, \quad t \in \mathbb{R},$$

Example. $M(z) \equiv \sigma|\Im z|$, $z \in \mathbb{C}$, where $\sigma > 0$ is a constant. Then $d\nu_M(t) = \dfrac{\sigma}{\pi}\,dt$, $t \in \mathbb{R}$.

Let $\Lambda = \bigl\{\lambda_k\bigr\}_{k\in\mathbb{N}} \subset \mathbb{C}$ be a point sequence without limit points in $\mathbb{C}$, i. e. $\lim\limits_{k\to\infty} \lambda_k = \infty$, and all points λ_k are pairwise various (*for simplicity*).

For $z \in \mathbb{C} \setminus \mathbb{R}$ *the Poisson integral* $\mathrm{P}\phi$ of function $\phi \in L^1(\mathbb{R}_*)$ is defined as

$$(\mathrm{P}\phi)(z) := \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{|\Im z|}{(t-\Re z)^2+(\Im z)^2}\,\phi(t)\,dt = \frac{1}{\pi}\int_{-\infty}^{+\infty} \left|\Im\frac{1}{t-z}\right|\,\phi(t)\,dt,$$

But for $x \in \mathbb{R}$ we set $(\mathrm{P}\phi)(x) := \phi(x)$. The *direct Hilbert transform* $\mathrm{H}\phi$ of ϕ is defined as

$$(\mathrm{H}\phi)(x) := \frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}_*} \frac{\phi(t)}{x-t}\,dt, \quad x \in \mathbb{R}_*,$$

where $\mathrm{PV}\int$ means the principal integral value in the sense of Cauchy.

Classes of the basic or test functions $R\mathcal{P}_0^m$, $2 \leq m \leq \infty$. This class will consist of all *positive functions* $\phi \colon \mathbb{R}_* \to [0,+\infty)$ from the class $C^m(\mathbb{R}_*)$ of m times continuously differentiable functions on $\mathbb{R}_*$ with

- *a finiteness condition* $\phi(x) \equiv 0$, $\quad |x| \geq R_\phi > 0$, where $R_\phi > 0$ is a constant;
- *a semi-normalization condition* $\limsup\limits_{x\to 0} \dfrac{\phi(x)}{\log\bigl(1/|x|\bigr)} \leq 1$;
- *a conjugate condition of positivity*

$$(-\mathrm{H}\phi)'(x) := \frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}_*} \frac{\phi(t)-\phi(x)}{(t-x)^2}\,dt \geq 0, \quad x \in \mathbb{R}_*,$$

  i. e. condition of increase for *the inverse Hilbert transform* $(\mathrm{H}\phi)^{-1} = -\mathrm{H}\phi$ separately on the negative semiaxis $(-\infty,0)$ and on the positive semiaxis $(0,+\infty)$.

Main Theorem. *Let $0 \notin \Lambda$. The following assertions are equivalent:*

(1) *there is an entire function $f \not\equiv 0$ such that $|f| \leq e^M$ on $\mathbb{C}$ and $f(\lambda_k) = 0$ for $k \in \mathbb{N}$;*

(2) *for the some (for any) $m \in \bigl(\mathbb{N} \setminus \{1\}\bigr) \cup \infty$ the condition*

$$\sup_{\phi\in R\mathcal{P}_0^m} \left(\sum_{k\in\mathbb{N}} (\mathrm{P}_{\mathbb{C}_\pm}\phi)(\lambda_k) - \int_{-\infty}^{+\infty} \phi(t)\,d\nu_M(t)\right) < +\infty$$

is fulfilled.

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